

# $G_2$ and the Rolling Ball

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## Abstract

Understanding the exceptional Lie groups as the symmetry groups of simpler objects is a long-standing program in mathematics. Here, we explore one famous realization of the smallest exceptional Lie group,  $G_2$ . Its Lie algebra  $\mathfrak{g}_2$  acts locally as the symmetries of a ball rolling on a larger ball, but only when the ratio of radii is 1:3. Using the split octonions, we devise a similar, but more global, picture of  $G_2$ : it acts as the symmetries of a ‘spinorial ball rolling on a projective plane’, again when the ratio of radii is 1:3. We explain this ratio in simple terms, use the dot product and cross product of split octonions to describe the  $G_2$  incidence geometry, and show how a form of geometric quantization applied to this geometry lets us recover the imaginary split octonions and these operations.

## 1 Introduction

When Cartan and Killing classified the simple Lie algebras, they uncovered five surprises: the exceptional Lie algebras. The smallest of these, the Lie algebra of  $G_2$ , was soon constructed explicitly by Cartan and Engel. However, it was not obvious how to understand this Lie algebra as arising from the symmetry group of a naturally occurring mathematical object. Giving a simple description of  $G_2$  has been a challenge ever since: though much progress has been made, the story is not yet finished.

In this paper, we study two famous realizations of the split real form of  $G_2$ , both essentially due to Cartan. First, this group is the automorphism group of an 8-dimensional nonassociative algebra: the split octonions. Second, it is roughly the group of symmetries of a ball rolling on a larger fixed ball without slipping or twisting, but *only when the ratio of radii is 1:3*.

The relationship between these pictures has been discussed before, and indeed, the history of this problem is so rich that we postpone all references to the next section, which deals with that history. We then explain how each description of  $G_2$  is hidden inside the other. On the one hand, a variant of the 1:3 rolling ball system, best thought of as a ‘spinor rolling on a projective plane’, lives inside the imaginary split octonions as the space of ‘light rays’: 1-dimensional null subspaces.

On the other hand, we can recover the imaginary split octonions from this variant of the 1:3 rolling ball via geometric quantization.

Using a spinorial variant of the rolling ball system may seem odd, but it is essential if we want to see the hidden  $G_2$  symmetry. In fact, we must consider three variants of the rolling ball system. The first is the ordinary rolling ball, which has configuration space  $S^2 \times \text{SO}(3)$ . This never has  $G_2$  symmetry. We thus pass to the double cover,  $S^2 \times \text{SU}(2)$ , where such symmetry is possible. We can view this as the configuration space of a ‘rolling spinor’: a rolling ball that does not come back to its original orientation after one full rotation, but only after two. To connect this system with the split octonions, it pays to go a step further, and identify antipodal points of the fixed sphere  $S^2$ . This gives  $\mathbb{RP}^2 \times \text{SU}(2)$ , which is the configuration space of a spinor rolling on a projective plane.

This last space explains why the 1:3 ratio of radii is so special. As mentioned, a spinor comes back to its original state only after two full turns. On the other hand, a point moving on the projective plane comes back to its original position after going halfway around the double cover  $S^2$ . Consider a ball rolling without slipping or twisting on a larger fixed ball. What must the ratio of their radii be so that the rolling ball makes two full turns as it rolls halfway around the fixed one? Or put another way: what must the ratio be so that the rolling ball makes four full turns as it rolls once around the fixed one? The answer is 1:3.

At first glance this may seem surprising. Isn’t the correct answer 1:4?

No: a ball of radius 1 turns  $R + 1$  times as it rolls once around a fixed ball of radius  $R$ . One can check this when  $R = 1$  using two coins of the same kind. As one rolls all the way about the other without slipping, it makes two full turns. Similarly, in our  $365\frac{1}{4}$  day year, the Earth actually turns  $366\frac{1}{4}$  times. This is why the sidereal day, the day as judged by the position of the stars, is slightly shorter than the ordinary solar day. The Earth is not rolling without slipping on some imaginary sphere. However, just as with the rolling ball, it makes an ‘extra turn’ by completing one full revolution around the center of its orbit.

Of course, this kind of reasoning only takes us so far. A spinor will come back to itself after any even number of turns, so the ratios of radii 1:3, 1:7, 1:11, and so on are all permitted by this argument—but only the first, 1:3, gives a system with  $G_2$  symmetry.

To understand this a bit better, we should bring the split octonions into the game. For any  $R > 1$  there is an incidence geometry with points and lines defined as follows:

- The points are configurations of a spinorial ball of radius 1 rolling on a fixed projective plane, with double cover a sphere of radius  $R$ .
- The lines are curves where the spinorial ball rolls along lines in the projective plane without slipping or twisting.

This space of points,  $\mathbb{RP}^2 \times \text{SU}(2)$ , is the same as the space of 1-dimensional null subspaces of the imaginary split octonions, which we call  $PC$ . Under this identification, the lines of our incidence geometry become certain curves in  $PC$ . If and only if  $R = 3$ , these curves ‘straighten out’: they are given by projectivizing certain 2-dimensional null subspaces of the imaginary split octonions. We prove this in Theorem 4.

Indeed, in the case of the 1:3 ratio, and *only* in this case, we can find the rolling ball system hiding inside the split octonions. A ‘null subalgebra’ of the split octonions is one where the product of any two elements is zero. In Theorem 5 we show that when  $R = 3$ , the above incidence geometry is isomorphic to one where:

- The points are 1d null subalgebras of the imaginary split octonions.
- The lines are 2d null subalgebras of the imaginary split octonions.

As a consequence, this geometry is invariant under the automorphism group of the split octonions: the split real form of  $G_2$ .

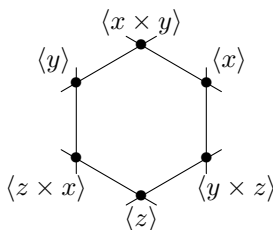
This group is also precisely the group that preserves the dot product and cross product operations on the imaginary split octonions. These are defined by decomposing the octonionic product into real and imaginary parts:

$$xy = -x \cdot y + x \times y,$$

where  $x \times y$  is an imaginary split octonion and  $x \cdot y$  is a real multiple of the identity, which we identify with a real number. One of our main goals here is to give a detailed description of the above incidence geometry in terms of these operations. The key idea is that any nonzero imaginary split octonion  $x$  with  $x \cdot x = 0$  spans a 1-dimensional null subalgebra  $\langle x \rangle$ , which is a point in this geometry. Given two points  $\langle x \rangle$  and  $\langle y \rangle$ , we say they are ‘at most  $n$  rolls away’ if we can get from one to the other by moving along a sequence of at most  $n$  lines. Then:

- $\langle x \rangle$  and  $\langle y \rangle$  are at most one roll away if and only if  $xy = 0$ , or equivalently,  $x \times y = 0$ .
- $\langle x \rangle$  and  $\langle y \rangle$  are at most two rolls away if and only if  $x \cdot y = 0$ .
- $\langle x \rangle$  and  $\langle y \rangle$  are always at most three rolls away.

We define a ‘null triple’ to be an ordered triple of nonzero null imaginary split octonions  $x, y, z$ , pairwise orthogonal, obeying the condition  $(x \times y) \cdot z = \frac{1}{2}$ . We show that any null triple gives rise to a configuration of points and lines like this:



In the theory of buildings, this sort of configuration is called an ‘apartment’ for the group  $G_2$ . Together with  $(x \times y) \times z$ , the six vectors shown here form a basis of the imaginary split octonions, as we show in Theorem 12. Moreover, we show in Theorem 13 that the split real form of  $G_2$  acts freely and transitively on the set of null triples.

We also show that starting from this incidence geometry, we can recover the split octonions using geometric quantization. The space of points forms a projective real variety,

$$PC \cong \mathbb{RP}^2 \times \mathrm{SU}(2).$$

There is thus a line bundle  $L \rightarrow PC$  obtained by restricting the dual of the canonical line bundle to this variety. Naively, one might try to geometrically quantize  $PC$  by forming the space of holomorphic sections of this line bundle. However, since  $PC$  is a *real* projective variety, and  $L$  is a *real* line bundle, the usual theory of geometric quantization does not directly apply. Instead we need a slightly more elaborate procedure where we take sections of  $L \rightarrow PC$  that extend to holomorphic sections of the complexification  $L^{\mathbb{C}} \rightarrow PC^{\mathbb{C}}$ . In Theorem 18 we prove the space of such sections is the imaginary split octonions. In Theorem 30, we conclude by using geometric quantization to reconstruct the cross product of imaginary split octonions, at least up to a constant factor.

## 2 History

On May 23, 1887, Wilhelm Killing wrote a letter to Friedrich Engel saying that he had found a 14-dimensional simple Lie algebra [2]. This is now called  $\mathfrak{g}_2$ . By October he had completed classifying the simple Lie algebras, and in the next three years he published this work in a series of papers [19].

Besides the already known classical simple Lie algebras, he claimed to have found six ‘exceptional’ ones. In fact he only gave a rigorous construction of the smallest,  $\mathfrak{g}_2$ . In his 1894 thesis, Élie Cartan [10] constructed all of them and noticed that two of them were isomorphic, so that there are really only five.

But already in 1893, Cartan had published a note [9] describing an open set in  $\mathbb{C}^5$  equipped with a 2-dimensional ‘distribution’—a smoothly varying field of 2d spaces of tangent vectors—for which the Lie algebra  $\mathfrak{g}_2$  appears as the infinitesimal symmetries. In the same year, in the same journal, Engel [15] noticed the same thing. As we shall see, this 2-dimensional distribution is closely related to the rolling ball. The point is that the space of configurations of the rolling ball is 5-dimensional, with a 2-dimensional distribution that describes motions of the ball where it rolls without slipping or twisting.

Both Cartan [11] and Engel [16] returned to this theme in later work. In particular, Engel discovered in 1900 that a generic antisymmetric trilinear form on  $\mathbb{C}^7$  is preserved by a group isomorphic to the complex form of  $G_2$ . Furthermore, starting from this 3-form he constructed a nondegenerate symmetric bilinear form on  $\mathbb{C}^7$ . This implies that the complex form of  $G_2$  is contained in a group isomorphic to  $SO(7, \mathbb{C})$ . He also noticed that the vectors  $x \in \mathbb{C}^7$  that are null—meaning  $x \cdot x = 0$ , where we write the bilinear form as a dot product—define a 5-dimensional projective variety on which  $G_2$  acts.

As we shall see, this variety is the complexification of the configuration space of a rolling spinorial ball on a projective plane. Furthermore, the space  $\mathbb{C}^7$  is best seen as the complexification of the space of imaginary octonions. Like the space of imaginary quaternions (better known as  $\mathbb{R}^3$ ), the 7-dimensional space of imaginary octonions comes with a dot product and cross product. Engel’s bilinear form on  $\mathbb{C}^7$  arises from complexifying the dot product. His antisymmetric trilinear form arises from the dot product together with the cross product via the formula  $x \cdot (y \times z)$ .

However, all this was seen only later. It was only in 1908 that Cartan mentioned that the automorphism group of the octonions is a 14-dimensional simple Lie group [12]. Six years later he stated something he probably had known for some time: this group is the compact real form of  $G_2$  [13].

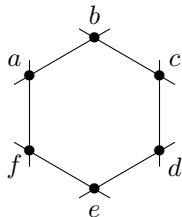
The octonions had been discovered long before, in fact the day after Christmas in 1843, by Hamilton’s friend John Graves. Two months before that, Hamilton had sent Graves a letter describing his dramatic discovery of the quaternions. This encouraged Graves to seek an even larger normed division algebra, and thus the octonions were born. Hamilton offered to publicize Graves’ work, but put it off or forgot until the young Arthur Cayley rediscovered the octonions in 1845 [14]. That this obscure algebra lay at the heart of *all* the exceptional Lie algebras became clear only slowly [4]. Cartan’s realization of its relation to  $\mathfrak{g}_2$ , and his later work on triality, was the first step.

In 1910, Cartan wrote a paper that studied 2-dimensional distributions in 5 dimensions [11]. Generically such a distribution is not integrable: the Lie bracket of two vector fields lying in this distribution does not again lie in this distribution. However, near a generic point, it lies in a 3-dimensional distribution. The Lie bracket of vector fields lying in this 3-dimensional distribution then generically give arbitrary tangent vectors to the 5-dimensional manifold. Such a distribution is called a ‘(2, 3, 5) distribution’. Cartan worked out a complete system of local geometric invariants for these distributions. He showed that if all these invariants vanish, the infinitesimal symmetries of a (2, 3, 5) distribution in a neighborhood of a point form the Lie algebra  $\mathfrak{g}_2$ .

Again this is relevant to the rolling ball. The space of configurations of a ball rolling on a surface is 5-dimensional, and it comes equipped with a (2, 3, 5) distribution. The 2-dimensional distribution describes motions of the ball where it rolls without twisting or slipping. The 3-dimensional distribution describes motions where it can roll and twist, but not slip. Cartan did not discuss rolling balls, but he does consider a closely related example: curves of constant curvature 2 or 1/2 in the unit 3-sphere.

Beginning in the 1950’s, François Bruhat and Jacques Tits developed a very general approach to incidence geometry, eventually called the theory of ‘buildings’ [3, 24], which among other things

gives a systematic approach to geometries having simple Lie groups as symmetries. In the case of  $G_2$ , because the Dynkin diagram of this group has two dots, the relevant geometry has two types of figure: points and lines. Moreover because the Coxeter group associated to this Dynkin diagram is the symmetry group of a hexagon, a generic pair of points  $a$  and  $d$  fits into a configuration like this, called an ‘apartment’:



There is no line containing a pair of points here except when a line is actually shown, and more generally there are no ‘shortcuts’ beyond what is shown. For example, we go from  $a$  to  $b$  by following just one line, but it takes two to get from  $a$  to  $c$ , and three to get from  $a$  to  $d$ .

For a nice introduction to these ideas, see the paper by Betty Salzberg [21]. Among other things, she notes that the points and lines in the incidence geometry of the split real form of  $G_2$  correspond to 1- and 2-dimensional null subalgebras of the imaginary split octonions. This was shown by Tits in 1955 [25].

In 1993, Robert Bryant and Lucas Hsu [8] gave a detailed treatment of curves in manifolds equipped with 2-dimensional distributions, greatly extending the work of Cartan. They showed how the space of configurations of one surface rolling on another fits into this framework. However, Igor Zelenko may have been the first to explicitly mention a ball rolling on another ball in this context, and to note that something special happens when their ratio of radii is 3 or  $1/3$ . In a 2005 paper [27], he considered an invariant of  $(2, 3, 5)$  distributions. He calculated it for the distribution arising from a ball rolling on a larger ball and showed it equals zero in these cases.

In 2006, Bor and Montgomery’s paper “ $G_2$  and the ‘rolling distribution’” put many of the pieces together [5]. They studied the  $(2, 3, 5)$  distribution on  $S^2 \times \text{SO}(3)$  coming from a ball of radius 1 rolling on a ball of radius  $R$ , and proved a theorem which they credit to Robert Bryant. First, passing to the double cover, they showed the corresponding distribution on  $S^2 \times \text{SU}(2)$  has a symmetry group whose identity component contains the split real form of  $G_2$  when  $R = 3$  or  $1/3$ . Second, they showed this action does not descend to original rolling ball configuration space  $S^2 \times \text{SO}(3)$ . Third, they showed that for any other value of  $R$  except  $R = 1$ , the symmetry group is isomorphic to  $\text{SU}(2) \times \text{SU}(2) / \pm(1, 1)$ . They also wrote:

Despite all our efforts, the ‘3’ of the ratio 1:3 remains mysterious. In this article it simply arises out of the structure constants for  $G_2$  and appears in the construction of the embedding of  $\mathfrak{so}(3) \times \mathfrak{so}(3)$  into  $\mathfrak{g}_2$ . Algebraically speaking, this ‘3’ traces back to the 3 edges in  $\mathfrak{g}_2$ ’s Dynkin diagram and the consequent relative positions of the long and short roots in the root diagram for  $\mathfrak{g}_2$  which the Dynkin diagram is encoding.

**Open problem.** Find a geometric or dynamical interpretation for the ‘3’ of the 3:1 ratio.

While Bor and Montgomery’s paper goes into considerable detail about the connection with split octonions, most of their work uses the now standard technology of semisimple Lie algebras: roots, weights and the like. In 2006 Sagerschnig [20] described the incidence geometry of  $G_2$  using the split octonions, and in 2008, Agrachev wrote a paper entitled “Rolling balls and octonions”. He emphasizes that the double cover  $S^2 \times \text{SU}(2)$  can be identified with the double cover of what we are calling  $PC$ , the projectivization of the space  $C$  of null vectors in the imaginary split octonions. He then shows that given a point  $\langle x \rangle \in PC$ , the set of points  $\langle y \rangle$  connected to  $\langle x \rangle$  by a single roll is the annihilator

$$\{x \in \mathbb{I} : yx = 0\}$$

where  $\mathbb{I}$  is the space of imaginary split octonions.

This sketch of the history is incomplete in many ways. For more details, try Agricola’s essay [2] on the history of  $G_2$  and Robert Bryant’s lecture about Cartan’s work on simple Lie groups of rank two [7]. Aroldo Kaplan’s review article “Quaternions and octonions in mechanics” is also very helpful [18]: it emphasizes the role that quaternions play in describing rotations, and the way an imaginary split octonion is built from an imaginary quaternion and a quaternion. We take advantage of this—and indeed most the previous work we have mentioned!—in what follows.

### 3 The rolling ball

Our goal is to understand  $G_2$  in terms of a rolling ball. It is *almost* true that split real form of  $G_2$  is the symmetry group of a ball of radius 1 rolling on a fixed ball 3 times as large without slipping or twisting. In fact we must pass to the double cover of the rolling ball system, but this is almost as nice: it is a kind of ‘rolling spinor’.

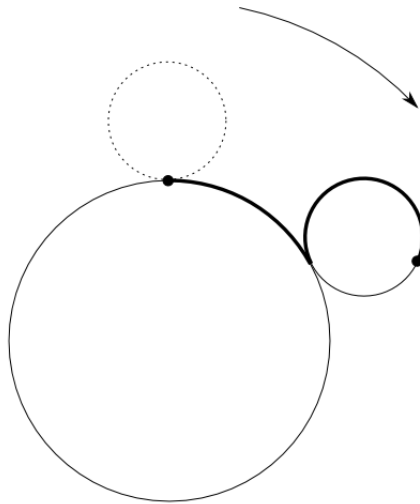
Before we talk about the rolling spinor, let us introduce the incidence geometry of the ordinary rolling ball. This differs from the usual approach to thinking of the rolling ball as a physical system with a constraint, but it is equivalent. There is an incidence geometry where:

- Points are configurations of a ball of radius 1 touching a fixed ball of radius  $R$ .
- Lines are trajectories of the ball of radius 1 rolling without slipping or twisting along great circles on the fixed ball of radius  $R$ .

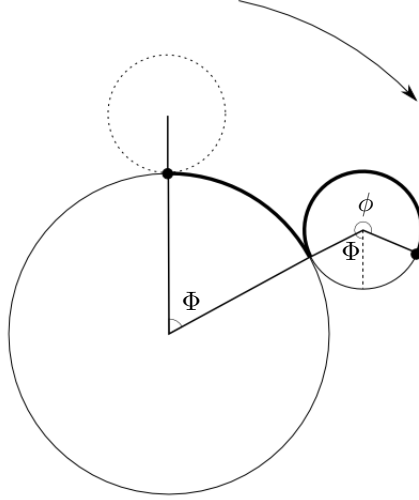
We call the ball of radius 1 the **rolling** ball, and the ball of radius  $R$  the **fixed** ball.

To specify a point in this incidence geometry, we can give a point  $x \in S^2$  on the unit sphere, together with a rotation  $g \in SO(3)$ . Physically,  $Rx \in \mathbb{R}^3$  is the point of contact where the rolling ball touches the fixed ball, while  $g$  tells us the orientation of the rolling ball, or more precisely how to obtain its orientation from some fixed, standard orientation. Thus we define the space of points in this incidence geometry to be  $S^2 \times SO(3)$ . This space is independent of the radius  $R$ , but the lines in this space depend on  $R$ . To see how we should define them, it helps to reason physically.

We begin with the assumption that, since the rolling ball is not allowed to slip or twist as it rolls, the point of contact traces paths of equal arclength on the fixed and rolling balls. In a picture:



Now let us quantify this. Begin with a configuration in which the rolling ball sits at the **north pole**,  $(0, 0, R) \in \mathbb{R}^3$ , of the fixed ball, and let it roll to a new configuration on a great circle passing through the north pole, sweeping out a central angle  $\Phi$  in the process. The point of contact thus traces out a path of arclength  $R\Phi$ . As the rolling ball turns, its initial point of contact sweeps out an angle of  $\phi$  relative to the line segment connecting the centers of both balls. In a picture:



By assumption, the distances traced out by the point of contact on the fixed and rolling balls are equal, and these are:

$$R\Phi = \phi,$$

since the rolling ball has unit radius. But because the frame of the rolling ball has itself rotated by angle  $\Phi$  in the frame of the fixed ball, the rolling ball has turned by an angle:

$$\phi + \Phi = (R + 1)\Phi.$$

So: in each revolution around the fixed ball, the rolling ball turns  $R + 1$  times! We urge the reader to check this directly for the case  $R = 1$  using two coins of the same sort. As one coin rolls around the other without slipping or twisting, it turns around twice.

This reasoning makes it natural to define the rolling trajectories using a parameterization. Let  $u$  and  $v$  be orthogonal unit vectors in  $\mathbb{R}^3$ . They both lie on  $S^2$ , and on the great circle parameterized by

$$\cos(\Phi)u + \sin(\Phi)v$$

where  $\Phi \in \mathbb{R}$ . If the rolling ball starts at  $u$  in the standard configuration, then when it rolls to  $\cos(\Phi)u + \sin(\Phi)v$ , it rotates about the axis  $u \times v$  by the angle  $(R + 1)\Phi$ . Writing  $\mathbf{R}(w, \alpha)$  for the right-handed rotation by an angle  $\alpha$  about the unit vector  $w$ , the rolling trajectory is

$$\{(\cos(\Phi)u + \sin(\Phi)v, \mathbf{R}(u \times v, (1 + R)\Phi)) : \Phi \in \mathbb{R}\} \subset S^2 \times \text{SO}(3).$$

More generally, the rolling ball may be rotated by some arbitrary element  $g \in \text{SO}(3)$  when it starts its trajectory. Then the rolling trajectory will be

$$L = \{(\cos(\Phi)u + \sin(\Phi)v, \mathbf{R}(u \times v, (1 + R)\Phi)g) : \Phi \in \mathbb{R}\} \subset S^2 \times \text{SO}(3) \quad (1)$$

We define a **line** in  $S^2 \times \text{SO}(3)$  to be any subset of this form. Of course this notion of line depends on  $R$ . Note that different choices of  $u, v$  and  $g$  may give different parametrizations of the same

line, since a rolling motion may start at any point along a given line. In fact the space of lines is 5-dimensional: two dimensions for the choice of our starting point  $u \in S^2$ , one dimension for the choice of  $v \in S^2$  orthogonal to  $u$ , determining the direction in which to roll, and three dimensions for the choice of starting orientation  $g \in \text{SO}(3)$ , minus one dimension of redundancy since our starting point on the line was arbitrary.

## 4 The rolling spinor

We now consider a situation where the rolling ball behaves like a spinor, in that it must make two whole turns instead of one to return to its original orientation. Technically this means replacing the rotation group  $\text{SO}(3)$  by its double cover, the group  $\text{SU}(2)$ . Since  $\text{SU}(2)$  can be seen as the group of unit quaternions, this brings quaternions into the game—and the split octonions follow soon after!

We begin with a lightning review of quaternions. Recall that the **quaternions**

$$\mathbb{H} = \{a + bi + cj + dk : a, b, c, d \in \mathbb{R}\}$$

form a real associative algebra with product specified by Hamilton's formula:

$$i^2 = j^2 = k^2 = ijk = -1.$$

The **conjugate** of a quaternion  $x = a + bi + cj + dk$  is defined to be  $\bar{x} = a - bi - cj - dk$ , and its **norm**  $|x|$  is defined by

$$|x|^2 = x\bar{x} = \bar{x}x = a^2 + b^2 + c^2 + d^2.$$

The quaternions are a **normed division algebra**, meaning that they obey

$$|xy| = |x||y|$$

for all  $x, y \in \mathbb{H}$ . This implies that the quaternions of norm 1 form a group under multiplication. This group is isomorphic to  $\text{SU}(2)$ , so indulging in a slight abuse of notation we simply write

$$\text{SU}(2) = \{q \in \mathbb{H} : |q| = 1\}.$$

Similarly, we can identify the **imaginary quaternions**

$$\text{Im}(\mathbb{H}) = \{x \in \mathbb{H} : \bar{x} = -x\}$$

with  $\mathbb{R}^3$ . The group  $\text{SU}(2)$  acts on  $\text{Im}(\mathbb{H})$  via conjugation: given  $q \in \text{SU}(2)$  and  $x \in \text{Im}(\mathbb{H})$ ,  $qxq^{-1}$  is again in  $\text{Im}(\mathbb{H})$ . This gives an action of  $\text{SU}(2)$  as rotations of  $\mathbb{R}^3$ , which exhibits  $\text{SU}(2)$  as a double cover of  $\text{SO}(3)$ .

We can now define a spinorial version of the rolling ball incidence geometry discussed in the last section. We define the space of points in the spinorial incidence geometry to be  $S^2 \times \text{SU}(2)$ . This is a double cover, and indeed the universal cover, of the space  $S^2 \times \text{SO}(3)$  considered in the previous section. So, we define a **line** in  $S^2 \times \text{SU}(2)$  to be the inverse image under the covering map

$$p: S^2 \times \text{SU}(2) \rightarrow S^2 \times \text{SO}(3)$$

of a line in  $S^2 \times \text{SO}(3)$ .

We can describe these lines more explicitly using quaternions:

**Proposition 1.** *Any line in  $S^2 \times \text{SU}(2)$  is of the form*

$$\tilde{L} = \{(e^{2\theta w}u, e^{(R+1)\theta w}q) : \theta \in \mathbb{R}\}.$$

for some orthogonal unit vectors  $u, w \in \text{Im}(\mathbb{H})$  and some  $q \in \text{SU}(2)$ .



*Proof.* First, remember Equation 1, which describes any line  $L \subset S^2 \times \text{SO}(3)$ :

$$L = \{(\cos(\Phi)u + \sin(\Phi)v, \mathbf{R}(u \times v, (1 + R)\Phi)g) : \Phi \in \mathbb{R}\} \subset S^2 \times \text{SO}(3)$$

in terms of orthogonal unit vectors in  $u, v \in \mathbb{R}^3$  and a rotation  $g \in \text{SO}(3)$ . To lift this line to  $S^2 \times \text{SU}(2)$ , we must replace the rotation  $g$  by a unit quaternion  $q$  that maps down to that rotation (there are two choices). Similarly, we must replace  $\mathbf{R}(u \times v, (1 + R)\Phi)$  by a unit quaternion that maps down to this rotation. The double cover  $\text{SU}(2) \rightarrow \text{SO}(3)$  acts as follows:

$$e^{\theta w/2} \mapsto \mathbf{R}(w, \theta)$$

for any unit vector  $w \in \text{Im}(\mathbb{H}) \cong \mathbb{R}^3$  and any angle  $\theta \in \mathbb{R}$ . Thus, the inverse image of the line  $L$  under the map  $p$  is

$$\tilde{L} = \{(\cos(\Phi)u + \sin(\Phi)v, e^{\frac{R+1}{2}\Phi(u \times v)}q) : \Phi \in \mathbb{R}\} \subset S^2 \times \text{SU}(2)$$

We can simplify this expression a bit by writing  $u \times v$  as  $w$ , so that  $u, v, w$  is a right-handed orthonormal triple in  $\text{Im}(\mathbb{H})$ . Then

$$\cos(\Phi)u + \sin(\Phi)v = e^{\frac{1}{2}\Phi w} u e^{-\frac{1}{2}\Phi w}$$

since this vector is obtained by rotating  $u$  by an angle  $\Phi$  around the axis  $w$ . However, since  $u$  and  $w$  are orthogonal imaginary quaternions, they anticommute, so we obtain

$$\cos(\Phi)u + \sin(\Phi)v = e^{\Phi w} u.$$

Thus any line in  $S^2 \times \text{SU}(2)$  is of the form

$$\tilde{L} = \{(e^{\Phi w} u, e^{\frac{R+1}{2}\Phi w} q) : \Phi \in \mathbb{R}\}$$

Even better, set  $\theta = \Phi/2$ . Then we have

$$\tilde{L} = \{(e^{2\theta w} u, e^{(R+1)\theta w} q) : \theta \in \mathbb{R}\}. \quad \square$$

## 5 The rolling spinor on a projective plane

We now consider a spinor rolling on a projective plane. In other words, we switch from studying lines on  $S^2 \times \text{SU}(2)$  to studying lines on  $\mathbb{RP}^2 \times \text{SU}(2)$ . As before, these lines depend on the radius  $R$  of the rolling ball.

There is a double cover

$$q: S^2 \times \text{SU}(2) \rightarrow \mathbb{RP}^2 \times \text{SU}(2)$$

Since  $S^2 \times \text{SU}(2)$  was introduced as a double cover of the  $S^2 \times \text{SO}(3)$  in the first place, it may seem perverse to introduce another space having  $S^2 \times \text{SU}(2)$  as a double cover:

$$\begin{array}{ccc} & S^2 \times \text{SU}(2) & \\ p \swarrow & & \searrow q \\ S^2 \times \text{SO}(3) & & \mathbb{RP}^2 \times \text{SU}(2) \end{array}$$

However,  $\mathbb{RP}^2 \times \text{SU}(2)$  is not diffeomorphic to the original rolling ball configuration space  $S^2 \times \text{SO}(3)$ . More importantly, it *is* diffeomorphic to the space of null lines through the origin in  $\text{Im}(\mathbb{H}) \oplus \mathbb{H}$ , a 7-dimensional vector space equipped with a quadratic form of signature  $(3, 4)$ .

To see this, first recall from Section 4 that a point in  $S^2 \times \text{SU}(2)$  is a pair  $(v, q)$  where  $v$  is a unit imaginary quaternion and  $q$  is a unit quaternion. So, a point in  $\mathbb{RP}^2 \times \text{SU}(2)$  is an equivalence class consisting of two points in  $S^2 \times \text{SU}(2)$ , namely  $(v, q)$  and  $(-v, q)$ . We write this equivalence class as  $(\pm v, q)$ .

We can describe a null line through the origin in  $\text{Im}(\mathbb{H}) \oplus \mathbb{H}$  in a very similar way. First, note that  $\text{Im}(\mathbb{H}) \oplus \mathbb{H}$  has a quadratic form  $Q$  given by

$$Q(a, b) = |a|^2 - |b|^2.$$

A **null vector** in this space is one with  $Q(x) = 0$ . Let  $C$  be the set of null vectors:

$$C = \{x \in \text{Im}(\mathbb{H}) \oplus \mathbb{H} : Q(x) = 0\}.$$

This is what physicists might call a **lightcone**. However, the signature of  $Q$  is  $(3, 4)$ , so this lightcone lives in an exotic spacetime with 3 time dimensions and 4 space dimensions.

Let  $PC$  be the corresponding **projective lightcone**:

$$PC = \{x \in C : x \neq 0\} / \mathbb{R}^*$$

where  $\mathbb{R}^*$ , the group of nonzero real numbers, acts by rescaling the cone  $C$ . A point in  $PC$  can be identified with a 1-dimensional **null subspace** of  $\text{Im}(\mathbb{H}) \oplus \mathbb{H}$ , by which we mean a subspace consisting entirely of null vectors. We can write any 1-dimensional null subspace as  $\langle x \rangle$ , the span of any nonzero null vector  $x$  lying in that subspace. We can always normalize  $x = (v, q)$  so that

$$|v|^2 = |q|^2 = 1.$$

The space of vectors  $x$  of this type is  $S^2 \times \text{SU}(2)$ , and two such vectors  $x$  and  $x'$  span the same subspace if and only if  $x' = \pm x$ . So, we shall think of a point in  $PC$  as an equivalence class of points in  $S^2 \times \text{SU}(2)$  consisting of the points  $(v, q)$  and  $(-v, -q)$ . We write this equivalence class as  $\pm(v, q)$ .

**Proposition 2.** *There is a diffeomorphism*

$$\tau: \mathbb{RP}^2 \times \text{SU}(2) \rightarrow PC$$

sending  $(\pm v, q)$  to  $\pm(v, vq)$ .

*Proof.* First note that  $\tau$  is well-defined: reversing the sign of  $v$  reverses the sign of  $(v, vq)$ . Next note that  $\tau$  has a well-defined inverse, sending  $\pm(v, q)$  to  $(\pm v, v^{-1}q)$ . It is easy to check that both  $\tau$  and its inverse are smooth.  $\square$

There is a double cover

$$q: S^2 \times \text{SU}(2) \rightarrow \mathbb{RP}^2 \times \text{SU}(2)$$

sending  $(v, q)$  to the equivalence class  $(\pm v, q)$ . We define a **line** in  $\mathbb{RP}^2 \times \text{SU}(2)$  to be the image of a line in  $S^2 \times \text{SU}(2)$  under this map  $q$ . We then define a **line** in  $PC$  to be the image of a line in  $\mathbb{RP}^2 \times \text{SU}(2)$  under the diffeomorphism  $\tau$ .

In short, we can think of configurations and trajectories of a rolling spinorial ball on a projective plane as points and ‘lines’ in  $PC$ . But this concept of ‘line’ depends on the radius  $R$  of the ball. When  $R = 3$ , these lines have a wonderful property: they come from projectivizing planes inside the lightcone  $C$ . To see this, we need an explicit description of these lines:

**Proposition 3.** *Fixing the radius  $R$ , every line in  $PC$  is of the form*

$$L = \{\pm(e^{2\theta w}u, e^{-(R-1)\theta w}uq) : \theta \in \mathbb{R}\} \subset PC.$$

for some orthogonal unit vectors  $u, w \in \text{Im}(\mathbb{H})$  and some  $q \in \text{SU}(2)$ .

*Proof.* Recall from Proposition 1 that any line in  $S^2 \times \text{SU}(2)$  is of the form

$$\{(e^{2\theta w} u, e^{(R+1)\theta w} q) : \theta \in \mathbb{R}\}$$

where  $u, w$  are orthogonal unit vectors in  $\text{Im}(\mathbb{H})$  and  $q \in \text{SU}(2)$ . Thus, any line in  $\mathbb{RP}^2 \times \text{SU}(2)$  is of the form

$$\{(\pm e^{2\theta w} u, e^{(R+1)\theta w} q) : \theta \in \mathbb{R}\}$$

and applying the map  $\tau$ , any line in  $PC$  is of the form

$$\{\pm(e^{2\theta w} u, e^{2\theta w} u e^{(R+1)\theta w} q) : \theta \in \mathbb{R}\}.$$

Since  $u$  and  $w$  are orthogonal imaginary quaternions, they anticommute, so we may rewrite this as

$$\{\pm(e^{2\theta w} u, e^{-(R-1)\theta w} u q) : \theta \in \mathbb{R}\}. \quad \square$$

Suppose  $X \subset \text{Im}(\mathbb{H}) \oplus \mathbb{H}$  is a 2-dimensional null subspace. Then we can projectivize it and get a curve in  $PC$ :

$$PX = \{x \in X : x \neq 0\} / \mathbb{R}^*.$$

When  $R = 3$ , and only then, every line in  $PC$  is a curve of this kind:

**Theorem 4.** *If and only if  $R = 3$ , every line in  $PC$  is the projectivization of a 2-dimensional null subspace of  $\text{Im}(\mathbb{H}) \oplus \mathbb{H}$ .*

*Proof.* To prove this, it helps to polarize  $Q$  and introduce a **dot product** on  $\text{Im}(\mathbb{H}) \oplus \mathbb{H}$ , namely the unique symmetric bilinear form such that

$$x \cdot x = Q(x).$$

A subspace  $X \subset \text{Im}(\mathbb{H}) \oplus \mathbb{H}$  is null precisely when this bilinear form vanishes on  $X$ . We will need an explicit formula for this bilinear form:

$$(a, b) \cdot (c, d) = a \cdot c - b \cdot d$$

where at right  $\cdot$  is the usual dot product on  $\mathbb{H}$ :

$$a \cdot b = \text{Re}(\bar{a}b).$$

We will also need to recall that the dot product of imaginary quaternions is the same as the usual dot product on  $\mathbb{R}^3$ .

Now consider an arbitrary line  $L \subset PC$ . By Proposition 3 this is of the form

$$L = \{\pm(e^{2\theta w} u, e^{-(R-1)\theta w} u q) : \theta \in \mathbb{R}\}$$

for some orthogonal unit vectors  $u, w \in \text{Im}(\mathbb{H})$  and  $q \in \text{SU}(2)$ . Assume that  $L$  is the projectivization of some null subspace  $X \subset \text{Im}(\mathbb{H}) \oplus \mathbb{H}$ . Then every pair of vectors  $x, y \in X$  must have  $x \cdot y = 0$ . We now show that this constrains  $R$  to equal 3.

Indeed, letting  $\theta = 0$ , one such vector is

$$x = (u, uq),$$

while letting  $\theta$  be arbitrary, another is

$$y = (e^{2\theta w} u, e^{(1-R)\theta w} uq).$$

We have

$$\begin{aligned} x \cdot y &= u \cdot e^{2\theta w} u - uq \cdot e^{(1-R)\theta w} uq \\ &= u \cdot e^{2\theta w} u - u \cdot e^{(1-R)\theta w} u \end{aligned}$$

where in the second step we note that right multiplication by a unit quaternion preserves the dot product. Since  $e^{2\theta w} u$  is  $u$  rotated by an angle  $2\theta$  about the  $w$  axis, which is orthogonal to  $u$ , we have

$$u \cdot e^{2\theta w} u = \cos(2\theta).$$

Similarly

$$u \cdot e^{(1-R)\theta w} u = \cos(2(1-R)\theta).$$

To ensure  $x \cdot y = 0$ , we thus need

$$\cos(2\theta) = \cos((1-R)\theta).$$

This must hold for all  $\theta$ , so we need  $1-R = \pm 2$ . Since we are assuming the rolling ball has positive radius, we conclude  $R = 3$ .

On the other hand, suppose that  $R = 3$ . Then any line in  $PC$  has the form:

$$L = \{\pm(e^{2\theta w} u, e^{-2\theta w} uq) : \theta \in \mathbb{R}\}.$$

Expanding the exponentials:

$$\begin{aligned} (e^{2\theta w} u, e^{-2\theta w} uq) &= (\cos(2\theta)u + \sin(2\theta)wu, \cos(2\theta)uq - \sin(2\theta)wuq) \\ &= \cos(2\theta)(u, uq) + \sin(2\theta)(wu, -wuq). \end{aligned}$$

we see that this vector lies in the 2-dimensional null subspace spanned by the orthogonal null vectors  $(u, uq)$  and  $(wu, -wuq)$ . Thus,  $L$  is the projectivization of a 2-dimensional null subspace.  $\square$

Assume  $R = 3$ . Then every line in  $PC$  is the projectivization of a 2-dimensional null subspace. But the converse is false: not every 2-dimensional null subspace gives a line in  $PC$  when we projectivize it. Which ones do? The answer requires us to introduce the split octonions! As we shall see in the next section, it is precisely the 2-dimensional ‘null subalgebras’ of the split octonions that give lines in  $PC$ .

## 6 Split octonions and the rolling ball

We have seen that the configuration space for a rolling spinorial ball on a projective plane is the projective lightcone  $PC$ . We have also seen that the lines in this space are especially nice when the ratio of radii is 1:3. To go further, we now identify  $\text{Im}(\mathbb{H}) \oplus \mathbb{H}$  with the imaginary split octonions. This lets us prove that when the ratio of radii is 1:3, lines in  $PC$  can be defined using the algebra structure of the split octonions. Thus, automorphisms of the split octonions act to give symmetries of the configuration space that map lines to lines. This symmetry group is  $G'_2$ , the split real form of  $G_2$ .

Every simple Lie group comes in a number of forms: up to covers, there is a unique complex form, as well as a compact real form and a split real form. Some groups have additional real forms: any real Lie group whose complexification is the complex form will do. For  $G_2$ , however, there are only the three forms. Each is the automorphism group of some 8-dimensional composition algebra—in other words, some form of the octonions.

A **composition algebra**  $A$  is a possibly nonassociative algebra with a multiplicative unit 1 and a nondegenerate quadratic form  $Q$  satisfying

$$Q(xy) = Q(x)Q(y)$$

for all  $x, y \in A$ . This concept makes sense over any field. Right now we only need real composition algebras, but in the next section we will need a complex one.

Up to isomorphism, there are just two 8-dimensional real composition algebras, and their automorphism groups give the two real forms of  $G_2$ :

- The **octonions**,  $\mathbb{O}$ , is the vector space  $\mathbb{H} \oplus \mathbb{H}$  with the product

$$(a, b)(c, d) = (ac - \bar{d}b, \bar{a}d + cb).$$

This becomes a composition algebra with the positive definite quadratic form given by

$$Q(a, b) = |a|^2 + |b|^2.$$

The automorphism group of  $\mathbb{O}$  is the compact real form of  $G_2$ , which we denote simply as  $G_2$ . This group is simply-connected and has trivial center.

- The **split octonions**,  $\mathbb{O}'$ , is the vector space  $\mathbb{H} \oplus \mathbb{H}$  with the product

$$(a, b)(c, d) = (ac + \bar{d}b, \bar{a}d + cb).$$

This becomes a composition algebra with the nondegenerate quadratic form of signature  $(4, 4)$  given by

$$Q(a, b) = |a|^2 - |b|^2.$$

The automorphism group of  $\mathbb{O}'$  is the split real form of  $G_2$ , which we denote as  $G'_2$ . More precisely, this is the adjoint split real form, which has fundamental group  $\mathbb{Z}_2$  and trivial center. There is also a simply-connected split real form with center  $\mathbb{Z}_2$ .

It is the split octonions,  $\mathbb{O}'$ , that are the most closely connected to the rolling ball. As with  $\mathbb{H}$ , the quadratic form on  $\mathbb{O}'$  can also be defined using conjugation. If we take

$$\overline{(a, b)} = (\bar{a}, -b).$$

then we can check that

$$Q(x) = x\bar{x} = \bar{x}x.$$

This conjugation satisfies some the same nice properties as quaternionic conjugation:

$$\overline{\bar{x}} = x, \quad \overline{\bar{x}y} = \bar{y}\bar{x}.$$

We define the **imaginary split octonions** by

$$\mathbb{I} = \{x \in \mathbb{O}' : \bar{x} = -x\} = \text{Im}(\mathbb{H}) \oplus \mathbb{H}.$$

Since conjugation in  $\mathbb{O}'$  is invariant under all the automorphisms of  $\mathbb{O}'$ , the same is true of the subspace  $\mathbb{I}$ , so we obtain a 7-dimensional representation of  $G'_2$ . This is well-known to be an irreducible representation. The quadratic form  $Q$  has signature  $(3, 4)$  when restricted to  $\mathbb{I}$ .

As promised at the start of this section, the lightcone in  $\text{Im}(\mathbb{H}) \oplus \mathbb{H}$  now lives in  $\mathbb{I}$ , the imaginary split octonions:

$$C \subset \mathbb{I}.$$

Moreover, because  $G'_2$  preserves the quadratic form on  $\mathbb{I}$ , it acts on  $C$ , as well as its projectivization:

$$PC = \{x \in C : x \neq 0\} / \mathbb{R}^*.$$

We have already seen how to view this space as the configuration space of a spinor rolling on a projective plane, and how to describe the rolling trajectories in that configuration space for any ratio of radii. We now show, when that ratio is 1:3, the action of  $G'_2$  preserves these rolling trajectories.

We define a **null subalgebra** of  $\mathbb{O}'$  to be a vector subspace  $V \subset \mathbb{O}'$  on which the product vanishes. In other words,  $V$  is closed under addition, scalar multiplication by real numbers, and  $xy = 0$  whenever  $x, y \in V$ . Such a subalgebra clearly does *not* contain the unit  $1 \in \mathbb{O}'$ . In fact, because the square of an element with nonzero real part cannot vanish, any null subalgebra must be purely imaginary. It must also be a null subspace of the imaginary split octonions, since  $Q(x) = x\bar{x} = -x^2 = 0$  for an imaginary split octonion in a null subalgebra. Thus, the projectivization of a null subalgebra gives a subset of the projective lightcone,  $PC$ .

**Theorem 5.** *Suppose  $R = 3$ . Then any line in  $PC$  is the projectivization of some 2d null subalgebra of  $\mathbb{O}'$ , and conversely, the projectivization of any 2d null subalgebra gives a line in  $PC$ .*

*Proof.* Let  $L$  be a line in  $PC$ . By Proposition 3 this is of the form

$$L = \{\pm(e^{2\theta w}u, e^{-2\theta w}uq) : \theta \in \mathbb{R}\} \subset PC$$

when  $R = 3$ . By Theorem 4,  $L$  is a projectivization of a 2-dimensional null subspace  $X \subset \text{Im}(\mathbb{H}) \oplus \mathbb{H}$ . This subspace is spanned by any two linearly independent vectors in  $X$ , so putting  $\theta = 0$  and  $\theta = \frac{\pi}{4}$  in our formula for  $L$ , we have:

$$X = \langle (u, uq), (wu, -wuq) \rangle.$$

We claim that  $X$  is a null subalgebra. To prove this, it suffices to check that the product of any two vectors in this basis vanishes. Because both vectors are null and imaginary, their squares automatically vanish:

$$Q(u, uq) = (u, uq)\overline{(u, uq)} = -(u, uq)^2 = 0,$$

and similarly for  $(wu, -wuq)$ . It thus remains to show that their product vanishes:

$$\begin{aligned} (u, uq)(wu, -wuq) &= (uwu + (-wuq)\bar{u}q, \bar{u}(-wuq) + wuuq) \\ &= (uwu - w, wuuq - wq) \\ &= 0, \end{aligned}$$

where we used the fact that the unit imaginary quaternion  $u$  anticommutes with  $w$ . Thus  $X$  is a 2-dimensional null subalgebra.

On the other hand, given a 2-dimensional null subalgebra  $X$ , we wish to show that its projectivization gives a line in  $PC$ . To prove this it suffices to show that  $X$  has the form

$$X = \langle (u, uq), (wu, -wuq) \rangle.$$

for some orthogonal unit imaginary quaternions  $u$  and  $w$  and unit quaternion  $q$ , since then reversing the calculation above shows that the projectivization of  $X$  is a curve in  $PC$  of this form:

$$L = \{\pm(e^{2\theta w}u, e^{-2\theta w}uq) : \theta \in \mathbb{R}\}.$$

So, fix any nonzero vector  $x \in X$ . It is easy to check that  $x = (u, uq)$  for some imaginary quaternion  $u$  and quaternion  $q$ . By rescaling, we can assume  $u$  has unit length, forcing  $q$  to also have unit length, since  $x$  is null.

Next choose any linearly independent vector  $y = (v, v') \in X$ . By subtracting a multiple of  $x$  from  $y$ , we can ensure the first component of  $y$  is orthogonal to the first component of  $x$ . By rescaling the result, we can also assume that  $v$  and  $v'$  both have unit length. We can thus obtain  $v$  from  $u$  by multiplication by a unit quaternion orthogonal to them both, say  $w$ :

$$v = wu.$$

In summary, we have:

$$X = \langle (u, uq), (wu, v') \rangle.$$

Finally, because  $X$  is a null subalgebra, we must have  $xy = 0$ , and this forces  $v' = -w u q$ . Indeed:

$$xy = (u, uq)(wu, v') = (uwu + v'\bar{u}q, \bar{u}v' + wuq),$$

and a quick calculation shows this vanishes if and only if  $v' = -w u q$ , as desired.  $\square$

**Corollary 6.** *When  $R = 3$ , the group  $G'_2$  acts on  $PC$  in a way that maps lines to lines.*

Knowing that the lines in  $PC$  correspond to 2-dimensional null subalgebras of the space  $\mathbb{I}$  of imaginary split octonions, we can use operations on  $\mathbb{I}$  to study the incidence geometry of  $PC$ . The concepts here we also apply to the complexification  $PC^{\mathbb{C}}$ , which we study in Section 9. Thus, we state them in a way that applies to both cases.

First, because the lines in  $PC$  are projectivizations of 2d null subalgebras, it will be very helpful for us to understand the **annihilator** of a null imaginary split octonion,  $x$ :

$$\text{Ann}_x = \{y \in \mathbb{I} : yx = 0\}.$$

This subspace of  $\mathbb{I}$  is intimately related to set of lines through  $\langle x \rangle \in PC$ : any point  $y \in \text{Ann}_x$  linearly independent of  $x$  will span a 2d null subalgebra with  $x$ , which in turn projectivizes to give a line through  $\langle x \rangle$ . So: understanding the annihilator is crucial for understanding how other points are connected to  $\langle x \rangle$  via lines, and this will move to the center of our focus in Section 9.

**Proposition 7.** *Let  $x \in C$  be a nonzero null vector. Then we have:*

1.  $\text{Ann}_x$  is a null subspace.
2. Any two elements of  $\text{Ann}_x$  anticommute.
3.  $\text{Ann}_x$  is three-dimensional.

*Proof.* First we show that  $\text{Ann}_x$  is a null subspace. Consider two elements  $y, y' \in \text{Ann}_x$ . In fact, because the dot product of two imaginary split octonions is proportional to their anticommutator:

$$y \cdot y' = -\frac{1}{2}(yy' + y'y),$$

we can show that  $y$  and  $y'$  are orthogonal and anticommute in one blow, proving parts 1 and 2.

Indeed, the real number  $-2(y \cdot y')$  vanishes if and only if its product with a nonzero vector vanishes. We consider its product with  $x$ , since  $y$  and  $y'$  annihilate  $x$  by definition:

$$-2(y \cdot y')x = (yy')x + (y'y)x = y(y'x) + y'(yx) + [y, y', x] + [y', y, x] = 0.$$

where  $[x, y, z] = (xy)z - x(zy)$  is the **associator**. The first two terms are zero because  $y$  and  $y'$  annihilate  $x$ . The last two terms cancel since the associator is antisymmetric in its three arguments, thanks to the fact that the split octonions are alternative [22].

To prove part 3 and show that  $\text{Ann}_x$  is 3-dimensional, write the imaginary split octonion  $x$  as a pair  $(u, q) \in \text{Im}(\mathbb{H}) \oplus \mathbb{H}$ . Since rescaling the null vector  $x$  does not change  $\text{Ann}_x$ , we may assume without loss of generality that it is normalized so that  $u\bar{u} = q\bar{q} = 1$ . We shall show that  $\text{Ann}_x$  is isomorphic to the vector space of imaginary quaternions,  $\text{Im}(\mathbb{H})$ . To do this, let  $y$  be any element of  $\text{Ann}_x$ , and write it as a pair  $(c, d)$ . Then

$$xy = (uc + d\bar{q}, \bar{u}d + cq).$$

This expression vanishes if and only if  $d = -ucq$ . Thus  $y = (c, -ucq)$ , and the map

$$f: \begin{array}{ccc} \text{Im}(\mathbb{H}) & \rightarrow & \text{Ann}_x \\ c & \mapsto & (c, -ucq) \end{array}$$

is an isomorphism of vector spaces.  $\square$

For some familiar geometries, such as that of a projective space, any two points are connected by a line. This is not true for PC, however. We can see this using the rolling ball description: as the ball rolls along a great circle from one point of contact to another, it rotates in a way determined by the constraint of rolling without slipping or twisting. If our initial and final configurations do not differ by this rotation, there is no way to connect them by a single rolling motion. In general we need multiple rolls to connect two configurations, so we give the following definition:

**Definition 8.** We say that two points  $a, b$  are **at most  $n$  rolls away** if there is a sequence of points  $a_0, a_1, \dots, a_n$  such that the  $a_0 = a$ ,  $a_n = b$ , and for any two consecutive points there is a line containing those two points. We say  $a$  and  $b$  are  **$n$  rolls away** if  $n$  is the *least* number for which they are at most  $n$  rolls away.

Note that because there is a line containing any point, if  $a$  and  $b$  are at most  $n-1$  rolls away, they are also at most  $n$  rolls away. The following basic facts hold both for PC and its complexification:

**Proposition 9.** *We have:*

0. Two points  $a$  and  $b$  are zero rolls away if and only if  $a = b$ .
1. Two points  $a$  and  $b$  are one roll away if and only if there is a line containing them but  $a \neq b$ .
2. Two points  $a$  and  $c$  are two rolls away if and only if there exists a unique point  $b$  such that:
  - there is a line containing  $a$  and  $b$ ,
  - there is a line containing  $b$  and  $c$ .

*Proof.* Part 0 is immediate from a careful reading of Definition 8. Part 1 then follows. For part 2, first suppose  $a$  and  $c$  are two rolls away. Since they are at least two rolls away, for some point  $b$  there is a line containing  $a$  and  $b$  and a line containing  $b$  and  $c$ . We must show the point  $b$  with this property is unique.

Suppose  $b'$  were another such point. Let us write  $a = \langle x \rangle$ ,  $b = \langle y \rangle$ ,  $b' = \langle y' \rangle$ , and  $c = \langle z \rangle$ . We know  $x, z \in \text{Ann}_y$ , since  $\langle x, y \rangle$  and  $\langle y, z \rangle$  are 2d null subalgebras: the 2d null subalgebras that projectivize to give the lines joining  $a$  and  $b$  and  $b$  and  $c$ . Now, if  $\langle x, y, z \rangle$  is itself two-dimensional, then we have:

$$\langle x, y \rangle = \langle x, y, z \rangle = \langle y, z \rangle,$$

whence  $x$  and  $z$  are contained in a 2d null subalgebra and  $a$  and  $c$ , connected by a line, are actually one roll apart. So we must have  $\langle x, y, z \rangle$  three-dimensional, and hence  $x, y$  and  $z$  are linearly independent. In fact, we must have:

$$\text{Ann}_y = \langle x, y, z \rangle,$$

since, by Proposition 7,  $\text{Ann}_y$  is three-dimensional.

Similarly,  $\text{Ann}_{y'} = \langle x, y', z \rangle$ . In particular, since annihilators are null subspaces by Proposition 7,  $y'$  is orthogonal to  $x$  and  $z$ . Moreover, since  $y$  and  $y'$  both annihilate  $x$ ,  $y$  and  $y'$  are also orthogonal. Thus,  $\langle x, y, y', z \rangle$  is null, but because the maximal dimension of a null subspace of the 7-dimensional space  $\mathbb{I}$  is three,  $y'$  must be a linear combination of the other vectors:

$$y' = \alpha x + \beta y + \gamma z.$$

Multiplying by  $x$ :

$$xy' = \gamma xz = 0.$$

We must have  $xz \neq 0$ , otherwise  $a$  and  $c$  are joined by the line obtained from the 2d null subalgebra  $\langle x, z \rangle$ , so this implies  $\gamma = 0$ . Similarly, because  $y'z = 0$ , we can conclude  $\alpha = 0$ . Thus  $y' = \beta y$ . In other words,  $b = \langle y \rangle = \langle y' \rangle = b'$ .



Conversely, suppose there exists a unique point  $b$  such that  $a$  and  $b$  lie on a line and  $b$  and  $c$  lie on a line. Then clearly  $a$  and  $c$  are at most two rolls away. Suppose they were at most one roll away. Then there would be a line containing  $a$  and  $c$ . There are infinitely many points on this line, contradicting the uniqueness of  $b$ . Thus,  $a$  and  $c$  are exactly two rolls away.  $\square$

Given nonzero  $x, y \in C$ , how can we tell how many rolls away  $\langle x \rangle$  is from  $\langle y \rangle$ ? We can use the dot product and cross product of imaginary split octonions. We have already defined the **dot product** of split octonions by polarizing the quadratic form  $Q$ :

$$x \cdot x = Q(x),$$

but on  $\mathbb{I}$  it is proportional to the anticommutator:

$$x \cdot y = -\frac{1}{2}(xy + yx),$$

as easily seen by explicit computation. Similarly, we define the **cross product** of imaginary split octonions to be half the commutator:

$$x \times y = \frac{1}{2}(xy - yx).$$

For  $x, y \in \mathbb{I}$  we have

$$xy = x \times y - x \cdot y$$

where  $x \times y$  is an imaginary split octonion and  $x \cdot y$  is a multiple of the identity.

**Theorem 10.** *Suppose that  $\langle x \rangle, \langle y \rangle \in PC$ . Then:*

1.  $\langle x \rangle$  and  $\langle y \rangle$  are at most one roll away if and only if  $xy = 0$ , or equivalently,  $x \times y = 0$ .
2.  $\langle x \rangle$  and  $\langle y \rangle$  are at most two rolls away if and only if  $x \cdot y = 0$ .
3.  $\langle x \rangle$  and  $\langle y \rangle$  are always at most three rolls away.

*Proof.* For part 1, first recall that by definition,  $\langle x \rangle$  is at most one roll away from  $\langle y \rangle$  if and only if  $\langle x, y \rangle$  is a null subalgebra. This happens if and only if  $xy = 0$  and  $yx = 0$ . But

$$yx = \bar{y}\bar{x} = \overline{xy}$$

since for the imaginary split octonions  $x$  and  $y$ , we have  $\bar{x} = -x$  and  $\bar{y} = -y$ . Thus, it is enough to say  $xy = 0$ .

Next let us show that  $xy = 0$  if and only if  $x \times y = 0$ . If  $xy = 0$ , then  $yx = 0$  as well by the above calculation, so  $x \times y$ , being half the commutator of  $x$  and  $y$ , is also zero.

For the converse, suppose  $x \times y = 0$ . Then  $x$  and  $y$  commute, so  $xy = -x \cdot y$ . Thus, it suffices to show  $x \cdot y = 0$ . Since  $x \neq 0$ , it is enough to show  $(x \cdot y)x = 0$ . For this we use the fact that the split octonions are **alternative**: the subalgebra generated by any two elements is associative [22]. The subalgebra generated by  $x$  and  $y$  is thus associative and commutative, so indeed

$$(x \cdot y)x = -\frac{1}{2}(xy + yx)x = -x^2y = (x \cdot x)y = 0$$

where in the last step we use the fact that  $x$  is null.

For part 2, first suppose that  $\langle x \rangle$  and  $\langle z \rangle$  are at most two rolls away. Then there is a point  $\langle y \rangle \in PC$  that is at most one roll away from  $\langle x \rangle$  and also from  $\langle z \rangle$ . Thus we know  $xy = 0 = zy$  by part 1. We wish to conclude that  $x \cdot z = 0$ . But this follows because  $x, z \in \text{Ann}_y$ , and annihilators are null subspaces by Proposition 7.

For the converse suppose  $x \cdot z = 0$ . If  $xz = 0$  we are done, since by part 1 it follows that  $x$  and  $z$  are at most one roll away. If  $xz \neq 0$  we can take  $\langle xz \rangle \in PC$ , and we claim this point is at most one roll away from  $\langle x \rangle$  and also from  $\langle z \rangle$ . To check this, by part 1 it suffices to show  $x(xz) = 0$  and  $(xz)z = 0$ . But since the split octonions are alternative, we have

$$x(xz) = x^2z = -(x \cdot x)z = 0$$

since  $x$  is null. Similarly  $(xz)z = 0$  since  $z$  is null.

For part 3, now let us show that every pair of points in  $PC$  is at most three rolls away. It suffices to show that given  $\langle x \rangle, \langle z \rangle \in PC$ , there exists  $\langle y \rangle$  that is at most one roll away from  $\langle x \rangle$  and at most two rolls away from  $\langle z \rangle$ . Thus, by parts 1 and 2, we need to find a nonzero null imaginary octonion  $y$  with  $xy = 0$  and  $y \cdot z = 0$ .

By Proposition 7, the space  $\text{Ann}_x$  of  $y$  with  $xy = 0$  is 3-dimensional. Thus the linear map:

$$\begin{aligned} \text{Ann}_x &\rightarrow \mathbb{R} \\ y &\mapsto y \cdot z \end{aligned}$$

has at least a two-dimensional kernel, guaranteeing the existence of the desired  $y$ . □

## 7 Null triples and incidence geometry

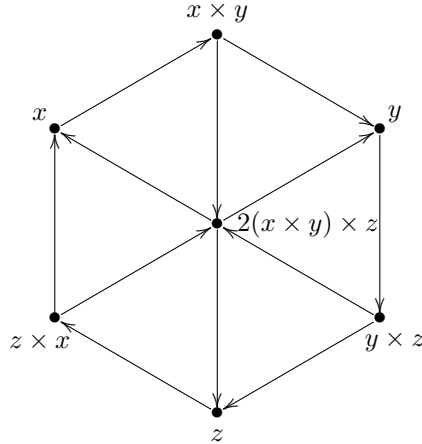
Next, we shall use our octonionic description of the rolling spinor to further investigate its incidence geometry. To do this, we introduce a tool we call a ‘null triple’.

**Definition 11.** A **null triple** is an ordered triple of nonzero null imaginary split octonions  $x, y, z \in \mathbb{I}$ , pairwise orthogonal, obeying the normalization condition:

$$(x \times y) \cdot z = \frac{1}{2}.$$

We shall show that any null triple generates  $\mathbb{I}$  under the cross product, so the action of an automorphism  $g \in G'_2$  of the split octonions is determined by its action on a null triple. In fact, in Theorem 13, we prove that the set of all null triples is a  **$G'_2$ -torsor**: given two null triples, there exists a unique element of  $G'_2$  carrying the first to the second.

Null triples are well suited to the incidence geometry of the rolling spinor because they are null, so that each member of the triple projectivizes to give a point of  $PC$ . We shall see that these points are all two rolls away from each other. The relationship to  $G_2$  runs deeper, however, as one can see by examining the cross product multiplication table we describe below, which is conveniently plotted as a hexagon with an extra vertex in the middle:



The resemblance to the weight diagram of the 7-dimensional irreducible representation  $\mathbb{I}$  of  $G'_2$  is no accident! Indeed, for any such decomposition into weight spaces, three nonadjacent vertices of the outer hexagon will be spanned by a null triple.

We begin by showing that we can use the above hexagon to describe both the dot and cross product in  $\mathbb{I}$  starting with a null triple. The arrows on this hexagon help us keep track of the cross product. For convenience, we speak of ‘vertices’ when we mean the basis vectors in  $\mathbb{I}$  corresponding to the seven vertices in this diagram. For the commutative dot product:

- All six outer vertices are null vectors.
- Opposite pairs of vertices have dot product  $\frac{1}{2}$ .
- Each outer vertex is orthogonal to all the others except its opposite.
- The vertex in the middle is orthogonal to all the outer vertices, but it is not null. Instead, its dot product with itself is  $-1$ .

As for the anticommutative cross product:

- The cross product of adjacent outer vertices is zero.
- For any two outer vertices that are neither adjacent nor opposite, their cross product is given by the outer vertex between them if they are multiplied in the order specified by the orientation of the arrows. For example,  $(z \times x) \times (x \times y) = x$ .
- The cross product of opposite outer vertices, multiplied in the order specified by the orientation, is half the vertex in the middle.
- The cross product of the vertex in the middle and an outer vertex gives that outer vertex if they are multiplied in the order specified by the orientation.

Now, let us prove these claims:

**Theorem 12.** *Given a null triple  $(x, y, z)$ , the following is a basis for  $\mathbb{I}$ :*

$$x, \quad y, \quad z, \quad x \times y, \quad y \times z, \quad z \times x, \quad 2(x \times y) \times z.$$

*In terms of this basis, the dot and cross product on  $\mathbb{I}$  take the form described above.*

*Proof.* We start by computing the dot product of outer vertices: that is, vectors corresponding to vertices on the outside of the hexagon. Then we compute the cross product of outer vertices. Next we compute the dot and cross product of all the outer vertices with the middle vertex, and the dot product of the middle vertex with itself. Finally, we verify that the above vectors are indeed a basis.

First, let us check that each outer vertex is a null vector. For  $x$ ,  $y$ , and  $z$  this is true by the definition of a null triple, so we need only check it for the other three. It suffices to consider  $x \times y$ . Since  $x$  and  $y$  are orthogonal we have  $x \times y = xy = -yx$ , so using the alternative law we have

$$(x \times y)(x \times y) = -(xy)(yx) = (x(y^2))x = 0.$$

This implies that the dot product of  $x \times y$  with itself vanishes.

Next we check that both the dot and cross product of two adjacent outer vertices vanishes. It suffices to consider  $x$  and  $x \times y = xy$ . Since  $x$  is null, the alternative law gives  $x(xy) = x^2y = 0$  as desired. So, the dot and cross product of adjacent outer vertices both vanish. In other words, by Theorem 10, they give points in  $PC$  that are one roll away or less.

Thus, by the same theorem, outer vertices that are not opposite give points in  $PC$  that are two rolls away or less. It follows from this theorem that such vertices are orthogonal.

It remains to compute the dot product of opposite outer vertices. By the definition of null triple, the opposite vertices  $z$  and  $x \times y$  have dot product  $\frac{1}{2}$ . Let us check that this forces other opposite vertices to also pair to  $\frac{1}{2}$ , for instance  $x$  and  $y \times z$ :

$$\begin{aligned}
-2(x \cdot (y \times z)) &= x(y \times z) + (y \times z)x \\
&= x(yz) + (yz)x \\
&= x(yz) - (zy)x \\
&= (xy)z - z(yx) - [x, y, z] - [z, y, x] \\
&= (x \times y)z + z(x \times y) \\
&= -2(z \cdot (x \times y)) \\
&= -1.
\end{aligned}$$

In the fifth line, we use the fact that the associator,  $[x, y, z] = (xy)z - x(yz)$ , is antisymmetric in its three arguments, thanks to alternativity [22]. A very similar calculation shows that the third opposite pair,  $y$  and  $z \times x$ , also have dot product  $\frac{1}{2}$ .

Next we turn to the cross product of outer vertices. We have already seen that adjacent outer vertices have vanishing cross product. For vertices that are neither adjacent nor opposite, we need to show their cross product gives the outer vertex between them if they are multiplied in the order specified by the orientation of the arrows. This is true by definition in three cases. The other three cases require a calculation. For example, consider the cross product of  $y \times z$  and  $z \times x$ :

$$\begin{aligned}
(y \times z) \times (z \times x) &= (yz)(zx) \\
&= (zy)(xz) \\
&= z(yx)z \\
&= z(y \times x)z \\
&= -z^2(x \times y) + z(z(y \times x) + (y \times x)z) \\
&= -2z(z \cdot (y \times x)) \\
&= z
\end{aligned}$$

where in the last step we use  $(x \times y) \cdot z = \frac{1}{2}$  and in the third step, we use a **Moufang identity**:

$$(zy)(xz) = z(yx)z.$$

which holds in any alternative algebra [22]. We can omit some parentheses here thanks to alternativity. Very similar calculations apply for other pairs of vertices that are neither opposite nor adjacent.

The cross product of opposite vertices equals half the vertex in the middle, if we multiply them in the correct order. That is, we claim:

$$(x \times y) \times z = (y \times z) \times x = (z \times x) \times y.$$

In fact, this follows from the following identity:

$$(u \times v) \times u = 0$$

when  $u$  and  $v$  are null and orthogonal. Before verifying this identity, we show how it implies the claim. Note that  $x + z$  is null and orthogonal to the null vector  $y$ . Thus, by the identity:

$$((x + z) \times y) \times (x + z) = 0.$$

Using bilinearity to expand this expression, we get:

$$(x \times y) \times x + (x \times y) \times z + (z \times y) \times x + (z \times y) \times z = 0.$$

The first and last terms vanish by the identity, implying:

$$(x \times y) \times z = (y \times z) \times x$$

as desired. A similar calculation shows this equals  $(z \times x) \times y$ .

Let us verify that  $(u \times v) \times u = 0$  when  $u$  and  $v$  are null and orthogonal. Since  $u$  and  $v$  are orthogonal, they anticommute. Thus  $u \times v = uv$ . By the definition of the cross product:

$$(uv) \times u = \frac{1}{2}((uv)u - u(uv)) = -u^2v = (u \cdot u)v = 0.$$

where we have made use of the anticommutativity of  $u$  and  $v$ , along with the fact that the split octonions are alternative, so that the subalgebra generated by any two elements is associative [22].

Next we compute the dot and cross product of each outer vertex with the middle vertex, which we shall call  $w$ :

$$w = 2(x \times y) \times z.$$

To do this, we claim that these vectors:

$$1, \quad i = z + (x \times y), \quad j = z - (x \times y), \quad k = i \times j = w$$

span a copy of the split quaternions in  $\mathbb{O}'$ . Before we verify this claim, let us show how it determines the dot and cross product of the outer vertices with  $w$ . In the split quaternions,  $k$  is orthogonal to both  $i$  and  $j$ , which happens if and only if:

$$w \cdot z = 0, \quad w \cdot (x \times y) = 0.$$

Moreover,  $k \times i = j$ , which happens if and only if:

$$w \times z = z, \quad w \times (x \times y) = -(x \times y),$$

The other cases work the same way, since we have shown that  $(x \times y) \times z$  is unchanged by cyclic permutations of the factors  $x$ ,  $y$  and  $z$ .

Now let us check the claim that the  $1$ ,  $i$ ,  $j$  and  $k$  as defined above really do span a copy of the split quaternions. We need only check that  $i^2 = -j^2 = -1$ , and that  $i$  and  $j$  anticommute, since it then follows that  $k = i \times j = ij$  anticommutes with  $i$  and  $j$  and squares to  $1$ . For  $i$ , we have:

$$\begin{aligned} i^2 &= (z + x \times y)^2 \\ &= z^2 + z(x \times y) + (x \times y)z + (x \times y)^2 \\ &= -2z \cdot (x \times y) \\ &= -1 \end{aligned}$$

where we have used the fact that  $z$  and  $x \times y$  are null and pair to  $\frac{1}{2}$ . A similar calculation shows  $j^2 = 1$ , and a further quick calculation shows that  $i$  and  $j$  are orthogonal, and hence anticommute as desired. As a bonus, we obtain the dot product of the middle vertex  $w$  with itself, because:

$$w \cdot w = -w^2 = -k^2 = -1.$$

Finally, let us verify that the seven vertices give a basis of  $\mathbb{I}$ . Because  $x$  and  $y$  both give points of  $PC$  one roll away from the point corresponding to  $x \times y$ , it follows from Proposition 7 that

$$\text{Ann}_{x \times y} = \langle x, x \times y, y \rangle.$$

It follows that these three vectors are linearly independent, since they span a 3d null subspace. Similarly:

$$\text{Ann}_z = \langle z \times x, z, y \times z \rangle.$$

Moreover, these annihilators must be complementary: any nonzero element of  $\text{Ann}_{x \times y} \cap \text{Ann}_z$  gives a point at most roll away from  $\langle x \times y \rangle$  and  $\langle z \rangle$ , contradicting the fact that by Theorem 10 these points are more than two rolls apart, since the cross product  $x \times y$  and  $z$  is nonzero. Thus the vector space spanned by these two annihilators must be 6-dimensional, and it remains to find a seventh vector independent of the six basis vectors already named. Since  $w = 2(x \times y) \times z$  is orthogonal to both  $\text{Ann}_{x \times y}$  and  $\text{Ann}_z$ , it is the seventh independent vector.  $\square$

After the hard work of proving the previous theorem, the next is a direct consequence:

**Theorem 13.** *The set of null triples is a  $G'_2$ -torsor: given two null triples  $(x, y, z)$  and  $(x', y', z')$ , there exists a unique element of  $g \in G'_2$  taking one to the other:*

$$(gx, gy, gz) = (x', y', z').$$

*Proof.* Because the action of  $G'_2$  preserves the dot and cross products, it takes null triples to null triples. Moreover, the action of  $g \in G'_2$  on  $\mathbb{I}$  is determined by its action on a null triple, since a null triple generates  $\mathbb{I}$ . Thus, there is at most one element of  $G'_2$  taking  $(x, y, z)$  to  $(x', y', z')$ . To see there is at least one such element, consider the linear map:

$$g: \mathbb{I} \rightarrow \mathbb{I}$$

which maps the basis obtained from  $(x, y, z)$ :

$$x, \quad y, \quad z, \quad x \times y, \quad y \times z, \quad z \times x, \quad 2(x \times y) \times z$$

to that obtained from  $(x', y', z')$ :

$$x', \quad y', \quad z', \quad x' \times y', \quad y' \times z', \quad z' \times x', \quad 2(x' \times y') \times z'.$$

By Theorem 12, this linear isomorphism preserves the dot and cross product. Thus  $g \in G'_2$ , as desired.  $\square$

**Proposition 14.** *Given any pair of null vectors  $x, y \in \mathbb{I}$  such that  $\langle x \rangle$  and  $\langle y \rangle$  are two rolls away, there is a null vector  $z \in \mathbb{I}$  such that  $(x, y, z)$  is a null triple.*

*Proof.* Recall that the vectors  $x, y$  and  $x \times y$  span a maximal null subspace:

$$V = \text{Ann}_{x \times y} = \langle x, x \times y, y \rangle.$$

Pick any maximal null subspace  $W$  complementary to  $V$ . The quadratic form  $Q$  must be nondegenerate when restricted to the direct sum  $V \oplus W \subset \mathbb{I}$ . Thus the map taking the dot product with  $x$ :

$$\begin{aligned} W &\rightarrow \mathbb{R} \\ w &\mapsto w \cdot x \end{aligned}$$

must have a two-dimensional kernel. Similarly, the map taking the dot product with  $y$ :

$$\begin{aligned} W &\rightarrow \mathbb{R} \\ w &\mapsto w \cdot y \end{aligned}$$

must also have a two-dimensional kernel. These kernels must be distinct, or else  $x$  and  $y$  are proportional, contradicting their linear independence. Thus, the subspace of  $W$  orthogonal to both  $x$  and  $y$ , the intersection of these two-dimensional kernels in the three-dimensional  $W$ , is one-dimensional. Let  $z \in W$  span this intersection. By choice,  $z$  is orthogonal to  $x$  and  $y$ , so it must have nonzero dot product with  $x \times y$ , because otherwise the pairing on  $V \oplus W$  would be degenerate. Thus:

$$(x \times y) \cdot z \neq 0.$$

By rescaling  $z$  if necessary, we obtain:

$$(x \times y) \cdot z = \frac{1}{2}. \quad \square$$

We can use the preceding proposition to create a null triple starting from any pair of null vectors, not just those whose projectivizations are two rolls away.

**Proposition 15.** *We have:*

0. *Any null vector  $x \in \mathbb{I}$  is the first vector of some null triple  $(x, y, z)$ .*
1. *Given any pair of null vectors  $w, x \in \mathbb{I}$  such that  $\langle w \rangle$  and  $\langle x \rangle$  are one roll away, there is a null triple  $(x, y, z)$  such that  $w = x \times y$ .*
2. *Given any pair of null vectors  $x, y \in \mathbb{I}$  such that  $\langle x \rangle$  and  $\langle y \rangle$  are two rolls away, there is a null vector  $z \in \mathbb{I}$  such that  $(x, y, z)$  is a null triple.*
3. *Given any pair of null vectors  $w, x \in \mathbb{I}$  such that  $\langle w \rangle$  and  $\langle x \rangle$  are three rolls away, there is a null triple  $(x, y, z)$  such that  $\langle w \rangle = \langle y \times z \rangle$ .*

*Proof.* We apply Proposition 14 repeatedly. To prove part 0, choose any  $y$  two rolls away from  $x$ . By Proposition 14, there exists a  $z$  such that  $(x, y, z)$  is a null triple. For part 1, choose  $\langle y \rangle$  one roll away from  $\langle w \rangle$  not lying on the line joining  $\langle w \rangle$  and  $\langle x \rangle$ . By choice,  $\langle x \rangle$  and  $\langle y \rangle$  are two rolls away, and Proposition 9 tells us that  $\langle w \rangle$  is the unique point at most one roll away from  $\langle x \rangle$  and  $\langle y \rangle$ . On the other hand,  $\langle x \times y \rangle$  is also at most one roll from both  $\langle x \rangle$  and  $\langle y \rangle$ . To see this, note from Theorem 10 that the  $x$  and  $y$  are orthogonal, and thus anticommute. Hence  $x \times y = xy = -yx$ , and a quick calculation shows that  $x \times y$  annihilates both  $x$  and  $y$  because  $x$  and  $y$  are null. By another application of Theorem 10, we conclude  $\langle x \times y \rangle$  is at most one roll from both  $\langle x \rangle$  and  $\langle y \rangle$ . From the uniqueness of  $\langle w \rangle$ , it follows that  $\langle w \rangle = \langle x \times y \rangle$ . Rescaling  $y$  if necessary, we have that  $w = x \times y$ . Since  $x$  and  $y$  are two rolls apart, Proposition 14 gives the result.

Part 2 is just a restatement of Proposition 14. Finally, for part 3, note that Theorem 10 implies  $w \cdot x \neq 0$ . Hence the linear map

$$\begin{aligned} \text{Ann}_w &\rightarrow \mathbb{R} \\ u &\mapsto u \cdot x \end{aligned}$$

has rank one, and a two-dimensional kernel. Let  $y$  and  $z$  be orthogonal vectors spanning this kernel. As in the proof of part 1,  $\langle y \rangle$  and  $\langle z \rangle$  are each one roll away from  $\langle w \rangle$ , but two rolls away from each other: if they were one roll apart,  $yz = 0$ , and  $\langle w, y, z \rangle$  would be a three-dimensional null subalgebra. Since the maximal dimension of a null subalgebra is two, we must have  $yz \neq 0$ . It now follows from the argument in part 1 that  $\langle w \rangle = \langle y \times z \rangle$ . Further,  $(x, y, z)$  are pairwise orthogonal by construction. We claim that  $(x \times y) \cdot z \neq 0$ , and so rescaling  $z$  if necessary,  $(x, y, z)$  is a null triple.

To check this last claim, note that  $\text{Ann}_w = \langle w, y, z \rangle$ . Moreover,  $\text{Ann}_x$  and  $\text{Ann}_w$  are complementary null subspaces, both of maximal dimension: any nonzero vector in their intersection would give a point that is one roll away from both  $\langle w \rangle$  and  $\langle x \rangle$ , contradicting our assumption that these points are three rolls away. The inner product restricts to a nondegenerate inner product on the direct sum  $\text{Ann}_w \oplus \text{Ann}_x$ . In particular, since  $x \times y \in \text{Ann}_x$  is orthogonal to itself and all vectors in  $\text{Ann}_x$ , it must have a nonvanishing inner product with some vector in  $\text{Ann}_w$ , or else the inner product would be degenerate. But  $x \times y$  is also orthogonal to  $w$  and  $y$ , thanks to Theorem 10:  $\langle x \times y \rangle$  is one roll away from  $\langle y \rangle$ , which is one roll away from  $\langle w \rangle$ , so  $\langle x \times y \rangle$  and  $\langle w \rangle$  are at most two rolls away. Thus, we must have  $(x \times y) \cdot z \neq 0$ , as desired.  $\square$

We can use null triples to decompose  $\text{PC} \times \text{PC}$  into its orbits under  $G'_2$ . There are precisely four:

**Theorem 16.** *Under the action of  $G'_2$ , the space of pairs of configurations,  $\text{PC} \times \text{PC}$ , decomposes into the following orbits:*

0.  $X_0 \subset PC \times PC$ , the space of pairs zero rolls away from each other. This is the diagonal set:

$$X_0 = \{(\langle x \rangle, \langle x \rangle) \in PC \times PC\}.$$

1.  $X_1 \subset PC \times PC$ , the space of pairs one roll away from each other:

$$X_1 = \{(\langle x \rangle, \langle y \rangle) \in PC \times PC : \langle x \rangle \neq \langle y \rangle, \quad x \times y = 0\}.$$

2.  $X_2 \subset PC \times PC$ , the space of pairs two rolls away from each other:

$$X_2 = \{(\langle x \rangle, \langle y \rangle) \in PC \times PC : x \times y \neq 0, \quad x \cdot y = 0\}.$$

3.  $X_3 \subset PC \times PC$ , the space of pairs three rolls away from each other:

$$X_3 = \{(\langle x \rangle, \langle y \rangle) \in PC \times PC : x \cdot y \neq 0\}.$$

*Proof.* In essence, we combine Theorem 13 with Proposition 15.

To prove part 0, let  $x$  and  $x'$  be two nonzero null vectors in  $C$ . We claim there is an element  $g \in G'_2$  such that  $x' = gx$ . Indeed, by Proposition 15, there are null triples  $(x, y, z)$  and  $(x', y', z')$ , so let  $g$  be the element of  $G'_2$  guaranteed by Theorem 13 taking the first null triple to the second. It follows that  $G'_2$  acts transitively on nonzero null vectors, on  $PC$ , and thus on the diagonal subset  $X_0$  of  $PC \times PC$ .

For part 1, let  $\langle w \rangle$  and  $\langle x \rangle$  be points of  $PC$  that are one roll away. By Proposition 15, there is a null triple  $(x, y, z)$  such that  $w = x \times y$ . If  $\langle w' \rangle$  and  $\langle x' \rangle$  are another pair of points that are one roll away, let  $(x', y', z')$  be a null triple such that  $w' = x' \times y'$ . Now let  $g \in G'_2$  carry one null triple to the other. We then have  $x' = gx$  and

$$w' = x' \times y' = gx \times gy = g(x \times y) = gw.$$

It follows that  $G'_2$  acts transitively on  $X_1$ .

For part 2, let  $\langle x \rangle$  and  $\langle y \rangle$  be two rolls away. By Proposition 15, there is a null triple  $(x, y, z)$ . If  $\langle x' \rangle$  and  $\langle y' \rangle$  is another pair of points that are two rolls away, there is a null triple  $(x', y', z')$ . Letting  $g \in G'_2$  take one null triple to the other, we immediately conclude that  $G'_2$  is transitive on  $X_2$ .

Finally, for part 3, let  $\langle w \rangle$  and  $\langle x \rangle$  be three rolls away. By Proposition 15, there is a null triple  $(x, y, z)$  such that  $w = \langle y \times z \rangle$ . If  $\langle w' \rangle$  and  $\langle x' \rangle$  is another pair of points three rolls away, and  $(x', y', z')$  a null triple such that  $w' = \langle y' \times z' \rangle$ , let  $g \in G'_2$  take one null triple to the other. Then  $x' = gx$ , and

$$\langle w' \rangle = \langle gy \times gz \rangle = \langle g(y \times z) \rangle = \langle gw \rangle.$$

It follows that  $G'_2$  acts transitively on  $X_3$ . □

## 8 Geometric quantization

Recall that  $C$  is the lightcone in the imaginary split octonions:

$$C = \{x \in \mathbb{I} : Q(x) = 0\}.$$

Let  $PC$  be the corresponding real projective variety in  $\mathbb{P}\mathbb{I}$ :

$$PC = \{x \in C : x \neq 0\} / \mathbb{R}^*.$$

This is called a ‘real projective quadric’. We write  $\langle x \rangle$  for the 1-dimensional subspace of  $\mathbb{I}$  containing the nonzero vector  $x \in \mathbb{I}$ . Then each point of  $PC$  can be written as  $\langle x \rangle$  for some nonzero  $x \in C$ .



The projectivized lightcone comes equipped with a real line bundle  $L \rightarrow PC$  whose fiber over the point  $\langle x \rangle$  consists of linear functionals on  $\langle x \rangle$ :

$$L_{\langle x \rangle} = \{f: \langle x \rangle \rightarrow \mathbb{R} : f \text{ is linear}\}.$$

In other words,  $L$  is the restriction to  $PC$  of the dual of the canonical line bundle on the projective space  $\mathbb{P}\mathbb{I}$ .

We would like to recover  $\mathbb{I}$  from this line bundle over the projectivized lightcone via some process of ‘geometric quantization’. However, this process is best understood for holomorphic line bundles over Kähler manifolds [17, 26]. So, we start by complexifying everything.

If we complexify the split octonions we obtain an algebra  $\mathbb{C} \otimes \mathbb{O}'$  over the complex numbers which is canonically isomorphic to the complexification of the octonions,  $\mathbb{C} \otimes \mathbb{O}$ . This latter algebra is called the ‘bioctonions’. So, we call the complexification of the split octonions the **bioctonions**, and denote it simply by  $\mathbb{O}^{\mathbb{C}}$ . The quadratic form  $Q$  on  $\mathbb{O}'$  extends to a complex-valued quadratic form on  $\mathbb{O}^{\mathbb{C}}$ , which we also denote as  $Q$ . This quadratic form makes  $\mathbb{O}^{\mathbb{C}}$  into a composition algebra. We can polarize  $Q$  to obtain the **dot product** on  $\mathbb{O}^{\mathbb{C}}$ , the unique symmetric bilinear form for which

$$x \cdot x = Q(x)$$

for all  $x \in \mathbb{O}^{\mathbb{C}}$ .

Complexifying the subspace  $\mathbb{I} \subset \mathbb{O}'$  gives a 7-dimensional complex subspace of  $\mathbb{O}^{\mathbb{C}}$ . We denote this as  $\mathbb{I}^{\mathbb{C}}$  and call its elements **imaginary bioctonions**. We define

$$C^{\mathbb{C}} = \{x \in \mathbb{I}^{\mathbb{C}} : Q(x) = 0\}.$$

This is what algebraic geometers would call a ‘complex quadric’. We define  $PC^{\mathbb{C}}$  to be the corresponding projective variety in the complex projective space  $\mathbb{P}\mathbb{I}^{\mathbb{C}}$ :

$$PC^{\mathbb{C}} = \{x \in C^{\mathbb{C}} : x \neq 0\}/\mathbb{C}^*.$$

This is a ‘complex projective quadric’.

If we now change notation slightly and write  $\langle x \rangle$  for the 1-dimensional *complex* subspace of  $\mathbb{I}^{\mathbb{C}}$  containing the nonzero vector  $x \in \mathbb{I}^{\mathbb{C}}$ , then each point of  $PC^{\mathbb{C}}$  is of the form  $\langle x \rangle$  for some nonzero  $x \in C^{\mathbb{C}}$ . The complex projective quadric  $PC^{\mathbb{C}}$  comes equipped with a holomorphic complex line bundle  $L^{\mathbb{C}} \rightarrow PC^{\mathbb{C}}$  whose fiber at  $\langle x \rangle$  consists of all complex linear functionals on  $\langle x \rangle$ :

$$L_{\langle x \rangle} = \{f: \langle x \rangle \rightarrow \mathbb{C} : f \text{ is linear}\}.$$

Since the real projective quadric  $PC$  is included in the complex projective quadric  $PC^{\mathbb{C}}$ , and similarly the total space of the line bundle  $L$  is included in the total space of the complex line bundle  $L^{\mathbb{C}}$ , we have a commutative diagram:

$$\begin{array}{ccc} L & \longrightarrow & L^{\mathbb{C}} \\ \downarrow & & \downarrow \\ PC & \longrightarrow & PC^{\mathbb{C}} \end{array}$$

Complex conjugation gives rise to a conjugate-linear map from  $\mathbb{I}^{\mathbb{C}} = \mathbb{C} \otimes \mathbb{I}$  to itself, whose set of fixed points is just  $\mathbb{I}$ . This in turn gives an antiholomorphic map from  $PC^{\mathbb{C}}$  to itself whose fixed points are just  $PC$ . This lifts to an antiholomorphic map from  $L^{\mathbb{C}}$  to itself whose fixed points are just  $L$ . So, we actually have a commutative diagram

$$\begin{array}{ccc} L & \longrightarrow & L^{\mathbb{C}} \\ \downarrow & & \downarrow \\ PC & \longrightarrow & PC^{\mathbb{C}} \end{array}$$

Now we shall show that every imaginary bioctonion gives a holomorphic section of  $L^{\mathbb{C}}$ , and that every holomorphic section of  $L^{\mathbb{C}}$  arises this way. Even better, every imaginary split octonion gives a section of  $L$  that extends to a holomorphic section of  $L^{\mathbb{C}}$ , and every section of  $L$  with this property arises that way.

First, note that every imaginary bioctonion  $w$  gives a section  $s_w$  of  $L^{\mathbb{C}}$  as follows. Evaluated at point  $\langle x \rangle$  of  $PC^{\mathbb{C}}$ ,  $s_w$  must be some linear functional on the span of  $x \in C$ . We define this linear functional so that it maps  $x$  to  $w \cdot x$ :

$$s_w(x) = w \cdot x.$$

It is easy to check that the section  $s_w$  is holomorphic. Moreover, every holomorphic section of  $L^{\mathbb{C}}$  arises this way:

**Theorem 17.** *A section of  $L^{\mathbb{C}}$  is holomorphic if and only if it is of the form  $s_w$  for some imaginary bioctonion  $w$ , which is then unique. Thus, the space  $\mathbb{I}^{\mathbb{C}}$  of imaginary bioctonions is isomorphic to the space of holomorphic sections of  $L^{\mathbb{C}}$  over  $PC^{\mathbb{C}}$ .*

*Proof.* This is a direct consequence of the Bott–Borel–Weil Theorem [6, 23]. Any finite-dimensional irreducible complex representation of a complex semisimple Lie group  $G$  arises as the space of holomorphic sections of a holomorphic line bundle over  $G/P$  for some parabolic subgroup  $P$ . In particular, the irreducible representation of  $G_2^{\mathbb{C}}$  on  $\mathbb{I}^{\mathbb{C}}$  arises as the space of holomorphic sections of  $L^{\mathbb{C}} \rightarrow PC^{\mathbb{C}}$ , where  $PC^{\mathbb{C}} \cong G_2^{\mathbb{C}}/P$  with  $P$  being the subgroup that fixes a 1d null subspace in  $\mathbb{I}^{\mathbb{C}}$ . This implies that sections of the form  $s_w$  are all the holomorphic sections of  $L^{\mathbb{C}}$ . Clearly different choices of  $w$  give different sections  $s_w$ .  $\square$

We can think of  $\mathbb{I}$  as a real subspace of  $\mathbb{I}^{\mathbb{C}}$ , and then each  $w \in \mathbb{I}$  gives a section  $s_w$  of  $L^{\mathbb{C}}$  using the same construction. However, since  $w \cdot x$  is real when  $w, x \in \mathbb{I}$ , restricting this section to  $PC \subset PC^{\mathbb{C}}$  actually gives a section of the real line bundle  $L$ . Moreover:

**Theorem 18.** *A section of  $L \rightarrow PC$  extends to a global holomorphic section of  $L^{\mathbb{C}} \rightarrow PC^{\mathbb{C}}$  if and only if it is of the form  $s_w$  for some imaginary split octonion  $w$ , which is then unique. Thus, the space  $\mathbb{I}$  of imaginary split octonions is isomorphic to the space of sections of  $L$  over  $PC$  that extend to holomorphic sections of  $L^{\mathbb{C}}$  over all of  $PC^{\mathbb{C}}$ .*

*Proof.* Suppose we have a section of  $L$  that extends to a global holomorphic section of  $L^{\mathbb{C}}$ . By Theorem 17, this holomorphic section of  $L^{\mathbb{C}}$  is of the form  $s_w$  for some imaginary bioctonion  $w$ . Its restriction to  $PC \subset PC^{\mathbb{C}}$  will lie in  $L$  if and only if  $w \cdot x$  is real for all  $x \in C$ . This is true if and only if  $w \in \mathbb{I}$ . Clearly different choices of  $w$  give different sections  $s_w$ .  $\square$

## 9 The cross product from quantization

It would be nice if we could use geometric quantization to recover not only the vector space of imaginary octonions, but also the octonions together with their algebra structure. Here we do this for the bioctonions. We already know from Theorem 17 that geometric quantization of the space  $PC^{\mathbb{C}}$  gives a vector space isomorphic to the imaginary bioctonions. Now we will use geometric quantization to equip this space with an operation that matches the **cross product** of imaginary bioctonions:

$$x \times y = \frac{1}{2}(xy - yx).$$

This, we claim, is enough to recover the bioctonions as an algebra.

To see this, note that we can write  $\mathbb{O}^{\mathbb{C}} = \mathbb{C} \oplus \mathbb{I}^{\mathbb{C}}$  where  $\mathbb{C}$  consists of complex multiples of the identity  $1 \in \mathbb{O}^{\mathbb{C}}$ . To describe the multiplication of bioctonions it is thus enough to say what happens when we multiply two imaginary bioctonions. But the product of two imaginary bioctonions obeys

$$xy = x \times y - x \cdot y$$

where  $x \times y$  is an imaginary bioctonion and  $x \cdot y$  is a multiple of the identity. Here the **dot product** arises from polarizing the quadratic form on the bioctonions:

$$x \cdot x = Q(x),$$

but on imaginary bioctonions it is also proportional to the anticommutator:

$$x \cdot y = -\frac{1}{2}(xy + yx).$$

All this is easy to check by explicit computation.

Thus, to describe the bioctonions as an algebra it is enough to describe the cross product and dot product of imaginary bioctonions. Explicitly, multiplication in  $\mathbb{O}^{\mathbb{C}} = \mathbb{C} \oplus \mathbb{I}^{\mathbb{C}}$  is given by

$$(\alpha, a)(\beta, b) = (\alpha\beta - a \cdot b, \alpha b + \beta a + a \times b).$$

But in fact, the dot product can be recovered from the cross product:

$$a \cdot b = -\frac{1}{6}\text{tr}(a \times (b \times \cdot))$$

where the right-hand side refers to the trace of the map

$$a \times (b \times \cdot): \mathbb{I}^{\mathbb{C}} \rightarrow \mathbb{I}^{\mathbb{C}}.$$

It is clear that some such formula should be true, since  $\mathbb{I}^{\mathbb{C}}$  is an irreducible representation of  $G_2^{\mathbb{C}}$ , the complex form of  $G_2$ , so any two invariant bilinear forms are proportional. The constant factor can thus be checked by computing the trace of the operator  $(a \times (a \times \cdot))$  for a single imaginary bioctonion  $a$ ; briefly, we get  $-6a \cdot a$  because there is 6-dimensional subspace orthogonal to  $a$ , on which this operator acts as multiplication by  $-a \cdot a$ .

In short, the whole algebra structure of the bioctonions can be recovered from the cross product of imaginary bioctonions. We can even define  $G_2^{\mathbb{C}}$  to be the group of linear transformations of the imaginary bioctonions that preserve the cross product.

Thus it is interesting to see if we can construct the cross product using geometric quantization. In fact we can. The procedure uses a ‘correspondence’ between the complex manifolds  $PC^{\mathbb{C}}$  and  $PC^{\mathbb{C}} \times PC^{\mathbb{C}}$ , which is a diagram like this:

$$\begin{array}{ccc} & S & \\ p \swarrow & & \searrow i \\ PC^{\mathbb{C}} & & PC^{\mathbb{C}} \times PC^{\mathbb{C}} \end{array}$$

where the maps  $p$  and  $i$  exhibit  $S$  as a complex submanifold embedded in  $(PC^{\mathbb{C}})^3$ . But in the case we shall consider,  $i$  by itself is already an embedding.

We can use  $p$  to pull the line bundle  $L^{\mathbb{C}}$  from  $PC^{\mathbb{C}}$  back to  $S$ . Then, since  $i$  is an embedding, we can push the resulting line bundle forward to the submanifold  $i(S) \subset PC^{\mathbb{C}} \times PC^{\mathbb{C}}$ . But there is another line bundle on  $PC^{\mathbb{C}} \times PC^{\mathbb{C}}$ : the **external tensor product** of the dual canonical bundle with itself,  $L^{\mathbb{C}} \boxtimes L^{\mathbb{C}}$ , whose fiber over any point  $(a, b) \in PC^{\mathbb{C}} \times PC^{\mathbb{C}}$  is  $L_a \otimes L_b$ . The bundle  $L^{\mathbb{C}} \boxtimes L^{\mathbb{C}}$  restricts to a bundle over  $i(S)$ . This is potentially different than the bundle obtained by pulling back  $L^{\mathbb{C}}$  along  $p$  and then pushing it forwards along  $i$ . In Proposition 27, however, we show these line bundles on  $i(S)$  can be identified.

Recall that imaginary bioctonions can be identified with sections of  $L^{\mathbb{C}}$ . By the construction described so far, we can take any such section, pull it back to  $S$ , push it forward to  $i(S)$ , and then think of it as a section of  $L^{\mathbb{C}} \boxtimes L^{\mathbb{C}}$  restricted to  $i(S)$ . In Proposition 28 we show this section extends

to all of  $PC^{\mathbb{C}} \times PC^{\mathbb{C}}$ . The result can be identified with an element in the tensor square of the space of imaginary bioctonions. All in all, this procedure gives rise to a linear map

$$\Delta: \mathbb{I}^{\mathbb{C}} \rightarrow \mathbb{I}^{\mathbb{C}} \otimes \mathbb{I}^{\mathbb{C}}.$$

This is a kind of ‘comultiplication’ of imaginary bioctonions. However, the space of imaginary bioctonions can be identified with its dual using the dot product. This gives a linear map

$$\Delta^*: \mathbb{I}^{\mathbb{C}} \otimes \mathbb{I}^{\mathbb{C}} \rightarrow \mathbb{I}^{\mathbb{C}}.$$

and in Theorem 30 we show that this is the cross product, at least up to a nonzero constant factor.

The most interesting fact about this whole procedure is that the correspondence

$$\begin{array}{ccc} & S & \\ p \swarrow & & \searrow i \\ PC^{\mathbb{C}} & & PC^{\mathbb{C}} \times PC^{\mathbb{C}} \end{array}$$

can be defined *using solely the incidence geometry of the projective lightcone  $PC^{\mathbb{C}}$* : in other words, using only points and lines in this space, and the relation of a point lying on a line.

To do this, we start by extending the concept of line from the  $PC$  to its complexification  $PC^{\mathbb{C}}$ . Following Theorem 5, we define a **line** in  $PC^{\mathbb{C}}$  to be the projectivization of a 2d null subalgebra of the bioctonions. This is also the concept of line implicit in the Dynkin diagram of  $G_2$ : in the theory of buildings, given any simple Lie group, each dot in its Dynkin diagram corresponds to a type of figure in a geometry having that group as symmetries [3]. The details for  $G_2$  are nicely discussed by Agricola [2].

Given this concept of line, we can describe the correspondence of complex manifolds that yields a geometric description of the bioctonion cross product. We begin by defining the manifold  $S$ .

First, recall from Definition 8 and Proposition 9 that two points  $a, b \in PC^{\mathbb{C}}$  are ‘one roll away’ if  $a \neq b$  but there is some line containing both  $a$  and  $b$ . We define  $S$  to be the subset of  $(PC^{\mathbb{C}})^3$  consisting of triples  $(a, b, c)$  for which  $b$  is the only point that is one roll away from both  $a$  and  $c$ . In this situation  $a$  and  $c$  are ‘two rolls away’, and we call  $b$  the **midpoint** of  $a$  and  $c$ . We shall soon see that if  $a = \langle x \rangle$  and  $c = \langle z \rangle$  are two rolls away, their midpoint is  $b = \langle x \times z \rangle$ . So, the cross product is hidden in the incidence geometry, and we can use geometric quantization to extract it.

Next we define  $p$  and  $i$ . The map  $p$  picks out the midpoint:

$$\begin{array}{ccc} p: & S & \rightarrow PC^{\mathbb{C}} \\ & (a, b, c) & \mapsto b. \end{array}$$

The map  $i$  picks out the other two points:

$$\begin{array}{ccc} i: & S & \rightarrow (PC^{\mathbb{C}})^2 \\ & (a, b, c) & \mapsto (a, c). \end{array}$$

Next we show that  $S$  is a complex manifold and  $i$  is an embedding. We also show that  $p$  makes  $S$  into the total space of a fiber bundle over  $PC^{\mathbb{C}}$ , though we will not need this fact. To get started, we must relate the geometry of  $PC^{\mathbb{C}}$  to operations on the space of imaginary split octonions:

**Theorem 19.** *Suppose that  $\langle x \rangle, \langle y \rangle \in PC^{\mathbb{C}}$ . Then:*

1.  $\langle x \rangle$  and  $\langle y \rangle$  are at most one roll away if and only if  $xy = 0$ , or equivalently,  $x \times y = 0$ .
2.  $\langle x \rangle$  and  $\langle y \rangle$  are at most two rolls away if and only if  $x \cdot y = 0$ .

3.  $\langle x \rangle$  and  $\langle y \rangle$  are always at most three rolls away.

*Proof.* The proof here is exactly like that of Theorem 10, so we omit it. In particular, like the split octonions, the bioctonions are an alternative algebra [22].  $\square$

We define the **annihilator** of a nonzero element  $x \in C^{\mathbb{C}}$  to be this subspace of  $\mathbb{I}^{\mathbb{C}}$ :

$$\text{Ann}_x = \{y \in \mathbb{I}^{\mathbb{C}} : xy = 0\}.$$

**Proposition 20.** *Given a point  $\langle x \rangle \in PC^{\mathbb{C}}$ , the set of points that are at most one roll away from  $\langle x \rangle$  is the projectivization of  $\text{Ann}_x$ .*

*Proof.* Theorem 19 says that  $\langle y \rangle \in PC^{\mathbb{C}}$  is at most one roll away from  $\langle x \rangle$  if and only if  $xy = 0$ .  $\square$

**Proposition 21.** *Suppose  $y \in C^{\mathbb{C}}$  is nonzero. Then  $\text{Ann}_y$  is a 3-dimensional null subspace of  $\mathbb{I}^{\mathbb{C}}$ , and any two elements of  $\text{Ann}_y$  anticommute.*

*Proof.* Consider two nonzero elements  $x, z \in \text{Ann}_y$ . They anticommute if they have vanishing dot product, since their anticommutator  $xz + zx$  is proportional to their dot product.

So, we need only show that  $\text{Ann}_y$  is null and 3-dimensional. Since  $G_2^{\mathbb{C}}$  acts transitively on  $PC^{\mathbb{C}}$ , it suffices to prove this for a single chosen  $y \in C^{\mathbb{C}}$ . We do the special case where  $y$  actually lies in  $\mathbb{I} \subset \mathbb{I}^{\mathbb{C}}$ . In this case, we know from Lemma 7 that  $\{x \in \mathbb{I} : yx = 0\}$  is a 3-dimensional null real subspace of  $\mathbb{I}$ . Since  $\text{Ann}_y$  is the complexification of this space, it is a 3-dimensional null complex subspace of  $\mathbb{I}^{\mathbb{C}}$ .  $\square$

Now we are ready to study the set  $S$ :

**Proposition 22.** *The set  $S$  is given by*

$$S = \{(\langle x \rangle, \langle y \rangle, \langle z \rangle) \in (PC^{\mathbb{C}})^3 : xy = 0 = yz, xz \neq 0\}.$$

*Proof.* By Theorem 19, the conditions say that  $\langle x \rangle$  is one roll away from  $\langle y \rangle$  and  $\langle y \rangle$  is one roll away from  $\langle z \rangle$  but  $\langle x \rangle$  is not one roll away from  $\langle z \rangle$ . This is a way of saying that  $\langle x \rangle$  is two rolls from  $\langle z \rangle$ , with  $\langle y \rangle$  as their midpoint, which is the condition for this triple of points to be in  $S$ .  $\square$

It is also useful to express the set  $S$  in terms of the cross product, as promised above:

**Proposition 23.** *Let  $(\langle x \rangle, \langle y \rangle, \langle z \rangle)$  be a point of  $S$ . Then  $\langle y \rangle = \langle x \times z \rangle$ .*

*Proof.* By Proposition 22, we know that  $xy = 0 = yz$  and  $xz \neq 0$ . Noting that  $y = xz$  is a solution to the first two equations because  $x$  and  $z$  are null, we must have  $\langle y \rangle = \langle xz \rangle$  because the midpoint is unique. So, it suffices to check that  $xz = x \times z$ . Since the cross product is half the commutator, this happens precisely when  $x$  and  $z$  anticommute. By Proposition 21,  $x$  and  $z$  indeed anticommute, because they both lie in the annihilator of  $y$ .  $\square$

**Proposition 24.** *The set  $S$  is given by*

$$S = \{(\langle x \rangle, \langle x \times z \rangle, \langle z \rangle) \in (PC^{\mathbb{C}})^3 : x \cdot z = 0, x \times z \neq 0\}.$$

*Proof.* By Theorem 19 the conditions here say that  $\langle x \rangle$  is two rolls away from  $\langle z \rangle$ , and we know from Proposition 23 that in this case their midpoint is  $\langle x \times z \rangle$ .  $\square$

**Proposition 25.** *We have:*

1.  $i(S) = \{(\langle x \rangle, \langle z \rangle) : x \cdot z = 0, x \times z \neq 0\}$ .
2.  $i(S)$  is a complex submanifold of  $PC^{\mathbb{C}} \times PC^{\mathbb{C}}$ .

3.  $S$  is a complex submanifold of  $(\mathbb{P}C^{\mathbb{C}})^3$ .

4.  $i: S \rightarrow \mathbb{P}C^{\mathbb{C}} \times \mathbb{P}C^{\mathbb{C}}$  is an embedding of  $S$  as a complex submanifold of  $\mathbb{P}C^{\mathbb{C}} \times \mathbb{P}C^{\mathbb{C}}$ .

*Proof.* Part 1 is clear from Proposition 24.

For Part 2, to show  $i(S)$  is a complex submanifold of  $\mathbb{P}C^{\mathbb{C}}$ , we show that its preimage under the quotient map

$$q: (C^{\mathbb{C}} - 0)^2 \rightarrow \mathbb{P}C^{\mathbb{C}} \times \mathbb{P}C^{\mathbb{C}}$$

is a submanifold. Let us call this preimage  $X$ :

$$X = q^{-1}(i(S)).$$

The set  $X$  is contained in the open set  $U$  on which the cross product is nonvanishing:

$$X \subset U = \{(x, z) \in (C^{\mathbb{C}} - 0)^2 : x \times z \neq 0\}.$$

This open set is a complex submanifold itself. On this open set, we can verify that the dot product is a map of constant rank:

$$\begin{aligned} f: U &\rightarrow \mathbb{C} \\ (x, z) &\mapsto x \cdot z. \end{aligned}$$

It follows that the preimage of zero under this map of constant rank,  $X = f^{-1}(0)$ , is a submanifold.

To check that  $f$  indeed has constant rank, we compute the rank of its derivative at the point  $(x, z) \in U \subset (C^{\mathbb{C}} - 0)^2$ :

$$\begin{aligned} f_*: T_{(x,z)}U &\rightarrow \mathbb{C} \\ (\dot{x}, \dot{z}) &\mapsto \dot{x} \cdot z + x \cdot \dot{z}. \end{aligned}$$

The linear map  $f_*$  has rank one if and only if it is nonzero. Here, since  $U$  is an open subset of  $(C^{\mathbb{C}} - 0)^2$ ,  $\dot{x}$  is a tangent vector to  $C^{\mathbb{C}}$  at the point  $x$ , which we can identify with the set of all vectors in the ambient vector space,  $\mathbb{I}^{\mathbb{C}}$ , that are orthogonal to  $x$ . Likewise,  $\dot{z}$  is in the set of all vectors orthogonal to  $z$ . The derivative  $f_*$  vanishes if and only if these sets are equal: if  $x$  is a nonzero multiple of  $z$ . But then their cross product will vanish, so such pairs are excluded from  $U$ . Thus  $f$  has constant rank on  $U$ .

Finally, because  $X$  is the inverse image of some set under the quotient map  $q$ , its image  $i(S)$  is also a submanifold. This completes the proof of part 2.

Part 3 follows because  $S$  is the graph of a holomorphic function

$$\begin{aligned} g: i(S) &\rightarrow \mathbb{P}C^{\mathbb{C}} \\ (\langle x \rangle, \langle z \rangle) &\mapsto \langle x \times z \rangle. \end{aligned}$$

The graph of this function is  $S$ , and is a submanifold of  $i(S) \times \mathbb{P}C^{\mathbb{C}}$ , and in turn this is a submanifold of  $(\mathbb{P}C^{\mathbb{C}})^3$ .

For part 4, note that  $i$  is an embedding of complex manifolds because the map from the graph of a holomorphic map to its domain is always an embedding.  $\square$

As a side-note, we have:

**Proposition 26.**  $p: S \rightarrow \mathbb{P}C^{\mathbb{C}}$  is a holomorphic fiber bundle.

*Proof.* Over any point  $\langle y \rangle \in \mathbb{P}C^{\mathbb{C}}$  the fiber of  $p$  is  $\text{PAnn}_y \times \text{PAnn}_y$  with those pairs of points that are not two rolls apart removed. For this we need to remove any pair that includes  $\langle y \rangle$ , as well as any pair on the diagonal subset,  $D$ . Thus the fiber is the complex manifold

$$p^{-1}\langle y \rangle = \text{PAnn}_y \times \text{PAnn}_y - \langle y \rangle \times \text{PAnn}_y - \text{PAnn}_y \times \langle y \rangle - D.$$

We leave the proof of local triviality as an exercise for the reader, since we will not be using this fact.  $\square$

Since  $i$  is a complex analytic diffeomorphism onto its image  $i(S)$ , we can push forward any holomorphic line bundle  $\Lambda$  on  $S$  to a holomorphic line bundle over  $i(S)$ , which we call  $i_*\Lambda$ . Thus, we obtain a holomorphic line bundle  $i_*p^*L^{\mathbb{C}}$  over  $i(S)$ . However, this is isomorphic to the line bundle  $L^{\mathbb{C}} \boxtimes L^{\mathbb{C}}$  restricted to  $i(S)$ .

To see this, note that the fiber of  $i_*p^*L$  over a point  $(\langle x \rangle, \langle z \rangle)$  of  $i(S)$  is  $L_{\langle x \times z \rangle}$ , the dual of the line  $\langle x \times z \rangle$  in  $\mathbb{I}^{\mathbb{C}}$ . On the other hand, the fiber  $L^{\mathbb{C}} \boxtimes L^{\mathbb{C}}|_{i(S)}$  is  $L_{\langle x \rangle} \otimes L_{\langle z \rangle}$ . Since the cross product gives a map:

$$\langle x \rangle \otimes \langle z \rangle \rightarrow \langle x \times z \rangle$$

dualizing yields a map:

$$\Theta_{(\langle x \rangle, \langle z \rangle)}: L_{\langle x \times z \rangle} \rightarrow L_{\langle x \rangle} \otimes L_{\langle z \rangle}.$$

This may seem like a deceptive trick, since in a moment will use  $\Theta$  to construct the cross product. However,  $\Theta$  can be characterized in other ways, at least up to a constant multiple:

**Proposition 27.** *The map*

$$\Theta: i_*p^*L^{\mathbb{C}} \rightarrow L^{\mathbb{C}} \boxtimes L^{\mathbb{C}}|_{i(S)}$$

*is an isomorphism of holomorphic line bundles that is equivariant with respect to the action of  $G_2^{\mathbb{C}}$ . Moreover, any other  $G_2^{\mathbb{C}}$ -equivariant map between these line bundles is a constant multiple of  $\Theta$ .*

*Proof.* The map  $\Theta$  is holomorphic by construction, and because the cross product  $x \times z$  is nonzero for  $(\langle x \rangle, \langle z \rangle) \in i(S)$ ,  $\Theta$  is an isomorphism on each fiber. Since everything used to construct  $\Theta$  is  $G_2^{\mathbb{C}}$ -equivariant,  $\Theta$  is as well.

Now let  $\Theta'$  be another  $G_2^{\mathbb{C}}$ -equivariant map:

$$\Theta': i_*p^*L^{\mathbb{C}} \rightarrow L^{\mathbb{C}} \boxtimes L^{\mathbb{C}}|_{i(S)}$$

To prove the claim that  $\Theta'$  is a constant multiple of  $\Theta$ , first recall that Theorem 16 states that the set of pairs in  $PC \times PC$  that are two rolls apart is an orbit of  $G_2'$ . This result is straightforward to generalize to the complexification, and so we conclude that the set of pairs in  $PC^{\mathbb{C}} \times PC^{\mathbb{C}}$  that are two rolls apart is an orbit of  $G_2^{\mathbb{C}}$ . This is the set  $i(S)$ , so  $G_2^{\mathbb{C}}$  acts transitively on  $i(S)$ .

Picking any point  $(a, c) \in i(S)$ , we have  $\Theta'_{(a,c)} = \alpha\Theta_{(a,c)}$  for some constant  $\alpha$ , since  $\Theta_{(a,c)}$  spans the one-dimensional space of maps between the one-dimensional fibers. Using the transitive action of  $G_2^{\mathbb{C}}$  and the equivariance of  $\Theta$  and  $\Theta'$ , we conclude  $\Theta' = \alpha\Theta$ .  $\square$

In what follows, we use  $\Gamma$  to denote the space of global holomorphic sections of a holomorphic line bundle over a complex manifold:

**Proposition 28.** *There is a linear map*

$$\Delta: \Gamma(L^{\mathbb{C}}) \rightarrow \Gamma(L^{\mathbb{C}} \boxtimes L^{\mathbb{C}})$$

*that is equivariant with respect to the action of  $G_2^{\mathbb{C}}$  and has the property that if  $\psi \in \Gamma(L^{\mathbb{C}})$ , then  $\Delta\psi$  extends  $\Theta i_*p^*\psi$  from  $i(S)$  to all of  $PC^{\mathbb{C}} \times PC^{\mathbb{C}}$ .*

*Proof.* For any point  $\langle y \rangle \in PC^{\mathbb{C}}$ ,  $\psi_{\langle y \rangle}$  is an element of  $L_{\langle y \rangle}$ , meaning a linear functional on the 1-dimensional subspace  $\langle y \rangle$ . By Theorem 17, there exists an imaginary bioctonion  $w$  such that  $\psi = s_w$ . In other words,  $\psi$  is determined by the fact that

$$\psi_{\langle y \rangle}: y \mapsto w \cdot y.$$

The fiber of  $p^*L^{\mathbb{C}}$  over  $(\langle x \rangle, \langle y \rangle, \langle z \rangle)$  is just  $L_{\langle y \rangle}$ , and by definition of the pullback,

$$(p^*\psi)_{(\langle x \rangle, \langle y \rangle, \langle z \rangle)}: y \mapsto w \cdot y.$$

However, by Proposition 23, we know  $\langle y \rangle = \langle x \times z \rangle$ . Pushing forward along  $i$ , we thus have

$$(i_* p^* \psi)_{(\langle x \rangle, \langle z \rangle)} : x \times z \mapsto w \cdot (x \times z).$$

Finally, applying  $\Theta$ , we obtain

$$(\Theta i_* p^* \psi)_{(\langle x \rangle, \langle z \rangle)} : x \otimes z \mapsto w \cdot (x \times z).$$

Since this formula makes sense for all pairs  $(\langle x \rangle, \langle z \rangle)$ , and not just those in  $i(S)$ , we see that  $\Theta i_* p^* \psi$  extends to a global section of  $L^{\mathbb{C}} \boxtimes L^{\mathbb{C}}$  over  $PC^{\mathbb{C}} \times PC^{\mathbb{C}}$ , given by

$$(\Delta \psi)_{(\langle x \rangle, \langle z \rangle)} : x \otimes z \mapsto w \cdot (x \times z).$$

This section  $\Delta \psi$  is holomorphic because  $\psi$  and the cross product are both holomorphic. Moreover,  $\Delta \psi$  depends linearly on  $\psi$ , and it is equivariant by construction.  $\square$

Using the canonical isomorphism

$$\Gamma(L^{\mathbb{C}} \boxtimes L^{\mathbb{C}}) \cong \Gamma(L^{\mathbb{C}}) \otimes \Gamma(L^{\mathbb{C}})$$

together with the isomorphism

$$\Gamma(L^{\mathbb{C}}) \cong \mathbb{I}^{\mathbb{C}}$$

given by Theorem 17, we can reinterpret  $\Delta$  as a linear map

$$\Delta : \mathbb{I}^{\mathbb{C}} \rightarrow \mathbb{I}^{\mathbb{C}} \otimes \mathbb{I}^{\mathbb{C}}.$$

Furthermore,  $\mathbb{I}^{\mathbb{C}}$  is canonically identified with its dual using the dot product of imaginary bioctonions. This allows us to identify the adjoint of  $\Delta$  with a linear map we call

$$\Delta^* : \mathbb{I}^{\mathbb{C}} \otimes \mathbb{I}^{\mathbb{C}} \rightarrow \mathbb{I}^{\mathbb{C}}.$$

**Proposition 29.** *The adjoint  $\Delta^* : \mathbb{I}^{\mathbb{C}} \otimes \mathbb{I}^{\mathbb{C}} \rightarrow \mathbb{I}^{\mathbb{C}}$  is the cross product.*

*Proof.* In the proof of Theorem 28 we saw that, up to a nonzero constant factor,  $\Delta$  sends the section  $s_w$  of  $L^{\mathbb{C}}$  to the section of  $L^{\mathbb{C}} \boxtimes L^{\mathbb{C}}$  given by

$$(\Delta s_w)_{(\langle x \rangle, \langle z \rangle)} : x \otimes z \mapsto w \cdot (x \times z).$$

This means that the adjoint of  $\Delta$  is the cross product.  $\square$

So far our construction may seem like ‘cheating’, since we used the cross product to define the map  $\Delta$  whose adjoint is the cross product. However, we now show that that *any* map with some of the properties of  $\Delta$  must give the cross product up to a constant factor:

**Theorem 30.** *Suppose*

$$\delta : \Gamma(L^{\mathbb{C}}) \rightarrow \Gamma(L^{\mathbb{C}} \boxtimes L^{\mathbb{C}})$$

*is any linear map that is equivariant with respect to the action of  $G_2^{\mathbb{C}}$ . Then identifying  $\Gamma(L^{\mathbb{C}})$  with the imaginary bioctonions, the adjoint*

$$\delta^* : \mathbb{I}^{\mathbb{C}} \otimes \mathbb{I}^{\mathbb{C}} \rightarrow \mathbb{I}^{\mathbb{C}}$$

*is the cross product up to a nonzero constant factor.*

*Proof.* By construction,  $\delta^*$  is an intertwining operator between representations of  $G_2^{\mathbb{C}}$ . However, any such intertwiner is a constant multiple of the cross product, because the tensor square of the 7-dimensional irreducible representation of  $G_2^{\mathbb{C}}$  contains that irreducible representation with multiplicity one.  $\square$



## 10 Conclusions

The final theorem above raises a question. What does our construction of the cross product of imaginary octonions mean in terms of the physics of a rolling ball? For a preliminary answer, we can naively imagine a section of  $L^{\mathbb{C}} \rightarrow PC^{\mathbb{C}}$  as a wavefunction describing the quantum state of a rolling ball. Then we can take such a quantum state, ‘duplicate’ it to get a quantum state of two new rolling balls of which the original one was the midpoint, and extend this to get a quantum state of an arbitrary pair of rolling balls. This procedure gives a linear map

$$\mathbb{I}^{\mathbb{C}} \rightarrow \mathbb{I}^{\mathbb{C}} \otimes \mathbb{I}^{\mathbb{C}}$$

whose adjoint is the cross product.

However, this account is at best only roughly correct. Since the real projective quadric  $PC$  is the configuration space of a rolling spinorial ball on a projective plane, we would expect to quantize this system by forming the cotangent bundle  $T^*PC$ , a symplectic manifold, and applying some quantization procedure to that. Instead we passed to the complexification  $PC^{\mathbb{C}}$ , which is in fact Kähler, and applied geometric quantization to that. Since there are neighborhoods of  $PC$  in  $PC^{\mathbb{C}}$  that are isomorphic as symplectic manifolds to neighborhoods of the zero section of  $T^*PC$ , we can loosely think of  $PC^{\mathbb{C}}$  as a way of modifying the cotangent bundle to make it compact. In a rough sense this amounts to putting a ‘speed limit’ on the motion of the rolling ball, making its Hilbert space of states finite-dimensional. However, it would be good to understand this more precisely. This might also clarify the physical significance, if any, of the real vector space  $\mathbb{I}$  obtained by placing an extra condition on the vectors in the space  $\mathbb{I}^{\mathbb{C}}$  obtained by geometrically quantizing  $PC^{\mathbb{C}}$ .

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