Classifying Spaces for Topological 2-Groups

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for online references, see:
http://math.ucr.edu/home/baez/barcelona/
Čech Cohomology for Bundles

If $G$ is a topological group and $M$ is a topological space, we can describe a principal $G$-bundle $P \to M$ using a Čech cocycle. This consists of an open cover $\mathcal{U} = \{U_i\}$ of $M$ together with transition functions

$$g_{ij} : U_i \cap U_j \to G$$

such that

$$g_{ij}(x) g_{jk}(x) = g_{ik}(x)$$

commutes for all $x \in U_i \cap U_j \cap U_k$. 
Two Čech cocycles define isomorphic bundles iff they are **cohomologous**, meaning there are functions

\[ f_i : U_i \rightarrow G \]

such that

\[
\begin{array}{ccc}
  f_i(x) & \downarrow & f_j(x) \\
  g_{ij}(x) & \downarrow & g'_{ij}(x) \\
  f_i(x) & \downarrow & f_j(x) \\
  g_{ij}(x) & \downarrow & g'_{ij}(x)
\end{array}
\]

commutes for all \( x \in U_i \cap U_j \).
The set of cohomology classes of Čech cocycles is called $\check{H}(\mathcal{U}, G)$. Taking the inverse limit as we refine the open cover, we obtain the (first) Čech cohomology of $M$ with coefficients in $G$:

$$\check{H}(M, G) = \lim_{\mathcal{U}} \check{H}(\mathcal{U}, G)$$

There is a bijection between $\check{H}(M, G)$ and the set of isomorphism classes of principal $G$-bundles over $M$. 
A Famous Old Theorem

Here is the result we’d like to categorify — a result first due to Milnor but polished by Steenrod, Segal, Milgram and May:

**Thm.** Let $G$ be a well-pointed topological group. Let $BG$, the **classifying space** of $G$, be the geometric realization of the nerve of $G$. Then for any paracompact Hausdorff space $M$, there is a bijection

$$[M, BG] \cong \check{H}(M, G)$$

(A topological group $G$ is **well-pointed** if $1 \in G$ has a neighborhood of which it is a deformation retract.)
**Topological 2-Groups**

**Defn.** A **2-group** is a category with a group of objects and a group of morphisms, such that all the category operations are group homomorphisms.

We draw the objects like this:

```
  g
• ──── •
```

and the morphisms like this:

```
  g  f  g'
• ┌─┐ └─┘ •
      ↑    ↓
```


We can multiply objects:

multiply morphisms:

and compose morphisms:
All three operations have a unit and inverses. All three are associative, so these are well-defined:

```
  ●────●────●────●
  ●        ●        ●
  ●        ●        ●
  ●        ●        ●
```

Finally, the **interchange law** holds:

```
  ●────●────●────●
  ●        ●        ●
  ●        ●        ●
  ●        ●        ●
```

is well-defined.
**Defn.** A **topological 2-group** is a 2-group with a topological group of objects and a topological group of morphisms, for which all the 2-group operations are continuous.

Two examples important in string theory:

- Any simply-connected compact simple Lie group $G$ gives a topological 2-group $\text{String}(G)$. 
Čech Cohomology for 2-Bundles

The Basic Idea: a Čech cocycle with coefficients in a topological 2-group $G$ is a recipe for building a ‘principal $G$-2-bundle’ over $M$ using ‘transition functions’. Two such 2-bundles will be ‘equivalent’ when their cocycles are cohomologous.

We won’t define ‘2-bundles’ here: see Toby Bartels’ thesis or the work of Baas, Bökstedt and Kro.

Instead, let’s go straight to Čech cohomology!
Let $\mathcal{U} = \{U_i\}$ be an open cover of a topological space $M$, and let $\mathcal{G}$ be a topological 2-group

**Defn.** A Čech cocycle with coefficients in $\mathcal{G}$ consists of the following data:

For each $x \in U_i \cap U_j$, an object $g_{ij}(x)$ in $\mathcal{G}$ depending continuously on $x$.

For each $x \in U_i \cap U_j \cap U_k$, a morphism $h_{ijk}(x)$ in $\mathcal{G}$ depending continuously on $x$ that fills in this triangle:
Finally, the $h_{ijk}$ must make these tetrahedra commute:
Defn. Two Čech cocycles \((g, h)\) and \((g', h')\) are cohomologous if we have the following data:

For each \(x \in U_i\), an object \(f_i(x)\) of \(G\) depending continuously on \(x\).

For each \(x \in U_i \cap U_j\), a morphism \(k_{ij}(x)\) in \(G\) depending continuously on \(x\) that fills in this square:
Finally, the $k_{ij}$ must make these prisms commute:
Defn. Let $\check{H}(\mathcal{U}, \mathcal{G})$ be the set of cohomology classes of Čech cocycles. We define the Čech cohomology of $M$ with coefficients in $\mathcal{G}$ to be the inverse limit as we refine the cover:

$$\check{H}(M, \mathcal{G}) = \lim_{\mathcal{U}} \check{H}(\mathcal{U}, \mathcal{G})$$
Categorifying That Famous Old Theorem

**Thm.** Suppose $G$ is a well-pointed topological 2-group and $M$ is a paracompact Hausdorff space admitting good covers. Then there is a bijection

$$\tilde{H}(M, G) \cong [M, B|G|]$$

where the topological group $|G|$ is the geometric realization of the nerve of $G$. So, we call $B|G|$ the **classifying space** of $G$.

(A topological 2-group $G$ is **well-pointed** if both the topological groups in its corresponding crossed module are well-pointed. An open cover is **good** if each nonempty finite intersection of open sets in the cover is contractible.)
A Corollary:  
Bundles vs. 2-Bundles

**Cor.** There is a 1-1 correspondence between:

- equivalence classes of principal $\mathcal{G}$-2-bundles over $X$
- elements of the Čech cohomology $\check{H}(M, \mathcal{G})$
- homotopy classes of maps $f : X \to B|\mathcal{G}|$
- elements of the Čech cohomology $\check{H}(M, |\mathcal{G}|)$
- isomorphism classes of principal $|\mathcal{G}|$-bundles over $X$. 
Characteristic Classes
for String($G$)-2-bundles

Now suppose $G$ is a compact simply-connected simple Lie group and $\text{String}(G)$ is its string 2-group:

**Thm.** There is a short exact sequence of topological groups

\[
1 \rightarrow K(\mathbb{Z}, 2) \rightarrow B|\text{String}(G)| \rightarrow^p G \rightarrow 1
\]

where $p$ is a fibration. This exhibits $B|\text{String}(G)|$ as the 3-connected cover of $G$. 
Matt Ando helped us show the following:

**Cor.** The homomorphism

\[ B|\text{String}(G)| \xrightarrow{p} G \]

gives an algebra homomorphism:

\[ H^*(BG, \mathbb{R}) \xrightarrow{p^*} H^*(B|\text{String}(G)|, \mathbb{R}) \]

This is onto, with kernel generated by the ‘second Chern class’ \( c_2 \in H^4(BG, \mathbb{R}) \).

So: the real characteristic classes of String\((G)\)-2-bundles are just like those of \( G \)-bundles, but with the second Chern class killed!