

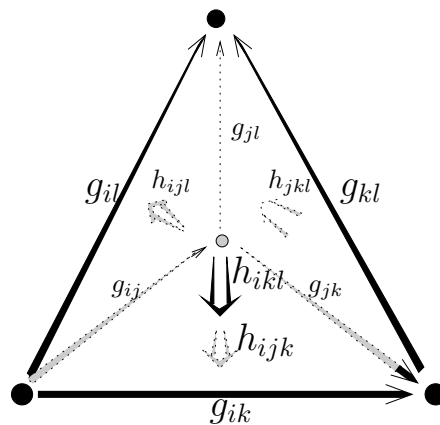
# Higher Gauge Theory – III

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joint work with:

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Notes and references at:

<http://math.ucr.edu/home/baez/barrett/>

# Bundles

First we'll recall some basic ideas from gauge theory; then we'll categorify them!

A **bundle** over a smooth space  $M$  is:

- a smooth space  $E$  (the **total space**),
- a smooth space  $F$  (the **fiber**),
- a smooth map  $p: E \rightarrow M$  (the **projection**),

such that  $M$  is covered by open sets  $U_i$  equipped with diffeomorphisms

$$t_i: p^{-1}U_i \rightarrow U_i \times F$$

(**local trivializations**) such that

$$\begin{array}{ccc} p^{-1}U_i & \xrightarrow{t_i} & U_i \times F \\ & \searrow p & \swarrow \\ & & U_i \end{array}$$

commutes.

In other words,  $E$  looks *locally* like the product of  $M$  and  $F$ ... but perhaps not *globally*.

# $G$ -Bundles

If  $F$  is a smooth space,  $\text{Aut}(F)$  is a smooth group. If  $E \rightarrow M$  is a bundle with fiber  $F$ , the local trivializations over open sets  $U_i$  covering  $M$  give smooth maps called **transition functions**:

$$g_{ij}: U_i \cap U_j \rightarrow \text{Aut}(F)$$

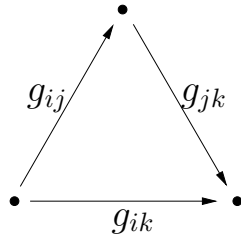
via:

$$t_j t_i^{-1}(x, f) = (x, g_{ij}(x)(f)).$$

These satisfy the **1-cocycle condition**

$$g_{ij}(x)g_{jk}(x) = g_{ik}(x)$$

for any  $x \in U_i \cap U_j \cap U_k$ . In other words, this diagram commutes:



For any smooth group  $G$ , we say the bundle  $E \rightarrow M$  has  $G$  as its **gauge group** when the maps  $g_{ij}$  factor through an action  $G \rightarrow \text{Aut}(F)$ . We then call  $E \rightarrow M$  a  **$G$ -bundle**.

Principal  $G$ -bundles over  $M$  are classified by the **1st Čech cohomology**,  $H^1(M, G)$ . For a good cover, this consists of 1-cocycles  $g_{ij}$  modulo coboundaries!

# Connections

Last time we treated holonomies as smooth functors

$$\text{hol}: \mathcal{P}_1(M) \rightarrow G$$

and showed these correspond to  $\mathfrak{g}$ -valued 1-forms  $A$  on  $M$ . Now this only works *locally!*

Suppose  $E \rightarrow M$  is a  $G$ -bundle with local trivializations over neighborhoods  $U_i$  covering  $M$ . Define a **connection** to be a smooth functor

$$\text{hol}_i: \mathcal{P}_1(U_i) \rightarrow G$$

for each  $i$ , such that the transition function  $g_{ij}$  defines a smooth natural isomorphism:

$$g_{ij}: \text{hol}_i|_{\mathcal{P}_1(U_i \cap U_j)} \rightarrow \text{hol}_j|_{\mathcal{P}(U_i \cap U_j)}$$

for all  $i, j$ . In other words, this diagram commutes:

$$\begin{array}{ccc}
 \bullet & \xrightarrow{g_{ij}(x)} & \bullet \\
 \text{hol}_i(\gamma) \downarrow & & \downarrow \text{hol}_j(\gamma) \\
 \bullet & \xrightarrow{g_{ij}(y)} & \bullet
 \end{array}$$

for any path  $\gamma: x \rightarrow y$  in  $U_i \cap U_j$ .

**Theorem.** There is a one-to-one correspondence between connections on the  $G$ -bundle  $E \rightarrow M$  and  $\mathfrak{g}$ -valued 1-forms  $A_i$  on the open sets  $U_i$  satisfying

$$A_i = g_{ij} A_j g_{ij}^{-1} + g_{ij} dg_{ij}^{-1}$$

on the intersections  $U_i \cap U_j$ .

## 2-Bundles

Now let's categorify all the above and get *higher gauge theory!* First we categorify the concept of *bundle*, following the thesis of Toby Bartels.

We can think of a smooth space  $M$  as a smooth category with only identity morphisms. A **2-bundle** over  $M$  consists of:

- a smooth category  $E$  (the **total space**),
- a smooth category  $F$  (the **fiber**),
- a smooth functor  $p: E \rightarrow M$  (the **projection**),

such that  $M$  is covered by open sets  $U_i$  equipped with smooth equivalences

$$t_i: p^{-1}U_i \rightarrow U_i \times F$$

(**local trivializations**) such that

$$\begin{array}{ccc} p^{-1}U_i & \xrightarrow{f} & U_i \times F \\ & \searrow p & \swarrow \\ & & U_i \end{array}$$

commutes.

# G-2-Bundles

**Theorem.** Let  $F$  be a smooth category and let  $\text{AUT}(F)$  be its automorphism 2-group, which is a smooth 2-group. Given a 2-bundle  $E \rightarrow M$  with fiber  $F$ , the local trivializations over open sets  $U_i$  covering  $M$  give:

- smooth maps

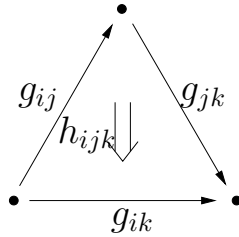
$$g_{ij}: U_i \cap U_j \rightarrow \text{Ob}(\text{AUT}(F))$$

- smooth maps

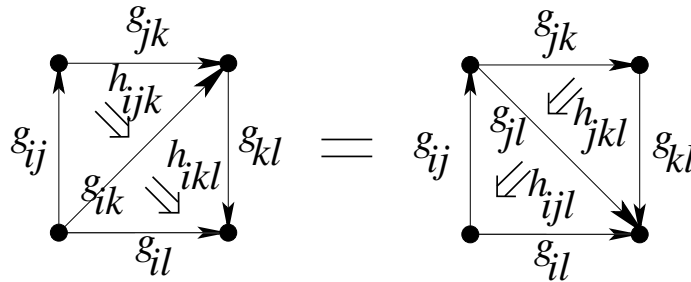
$$h_{ijk}: U_i \cap U_j \cap U_k \rightarrow \text{Mor}(\text{AUT}(F))$$

with

$$h_{ijk}(x): g_{ij}(x)g_{jk}(x) \rightarrow g_{ik}(x)$$

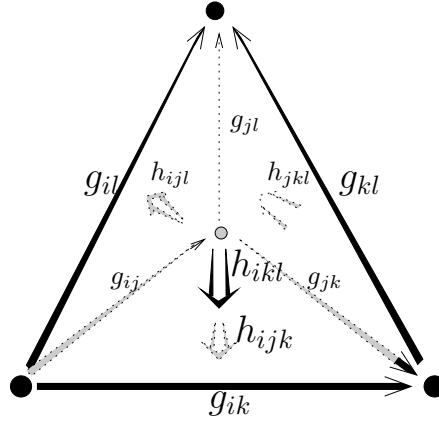


satisfying the **nonabelian 2-cocycle condition**:



on any quadruple intersection  $U_i \cap U_j \cap U_k \cap U_\ell$ .

In other words, this diagram commutes:



We say a 2-bundle  $E \rightarrow M$  has a smooth 2-group  $\mathcal{G}$  as its **gauge 2-group** when  $g_{ij}$  and  $h_{ijk}$  factor through an action  $\mathcal{G} \rightarrow \text{AUT}(F)$ . We then call  $E \rightarrow M$  is a  **$\mathcal{G}$ -2-bundle**.

**Theorem (Bartels).** Principal  $\mathcal{G}$ -2-bundles over  $M$  are classified by the 2nd Čech cohomology,  $H^2(M, \mathcal{G})$ . For a good cover, this consists of 2-cocycles  $(g_{ij}, h_{ijk})$  modulo coboundaries.

**Corollary.** Let  $\mathcal{G}$  be the smooth 2-group with one object and  $U(1)$  as morphisms. Then principal  $\mathcal{G}$ -2-bundles over  $M$  are classified by  $H^3(M, \mathbb{Z})$ .

**Corollary.** Let  $\mathcal{G} = \text{AUT}(H)$  for some smooth group  $H$ . Then the 2-category of  $\mathcal{G}$ -2-bundles over  $M$  is equivalent to the 2-category of nonabelian  $H$ -gerbes over  $M$ .

## 2-Connections

Let  $\mathcal{G}$  be a smooth 2-group and let  $E \rightarrow M$  be a  $\mathcal{G}$ -2-bundle equipped with local trivializations over open sets  $U_i$  covering  $M$ . Then a **2-connection** on  $E$  consists of the following data:

- For each  $i$  a smooth 2-functor:

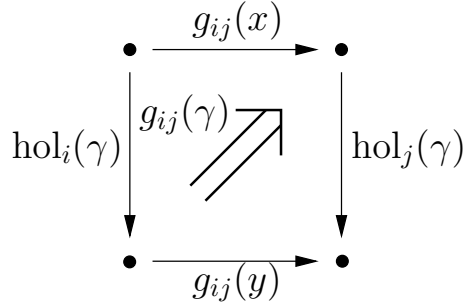
$$\begin{array}{ccc}
 \text{hol}_i : & \mathcal{P}_2(U_i) & \rightarrow & \mathcal{G} \\
 & \begin{array}{ccc} & \gamma & \\ x \curvearrowright & \Downarrow \Sigma & \curvearrowleft y \\ & \eta & \end{array} & \mapsto & \begin{array}{ccc} & \text{hol}_i(\gamma) & \\ \bullet \curvearrowright & \Downarrow \text{hol}_i(\Sigma) & \curvearrowleft \bullet \\ & \text{hol}_i(\eta) & \end{array}
 \end{array}$$



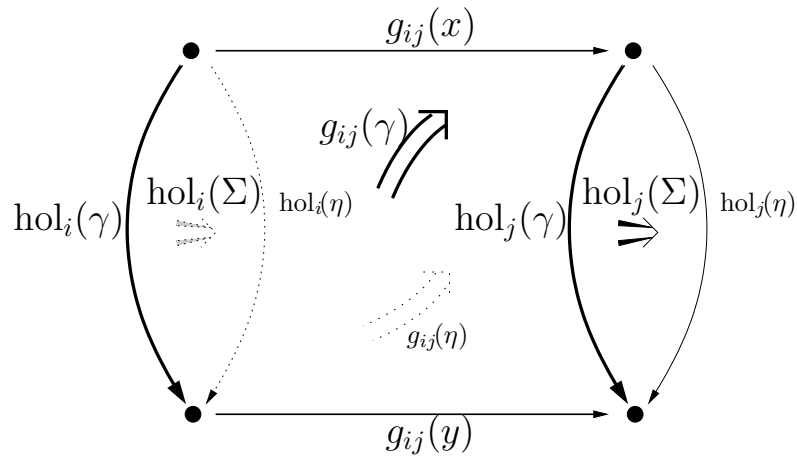
- For each  $i, j$  a pseudonatural isomorphism:

$$g_{ij}: \text{hol}_i|_{\mathcal{P}(U_i \cap U_j)} \rightarrow \text{hol}_j|_{\mathcal{P}(U_i \cap U_j)}$$

extending the transition function  $g_{ij}$ . In other words, for each path  $\gamma: x \rightarrow y$  in  $U_i \cap U_j$  a morphism in  $\mathcal{G}$ :

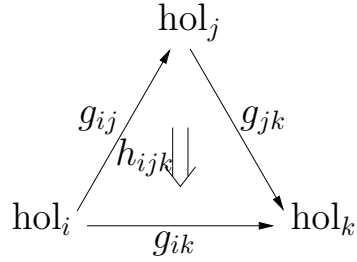


depending smoothly on  $\gamma$ , such that this diagram commutes for any surface  $\Sigma: \gamma \Rightarrow \eta$  in  $U_i \cap U_j$ :

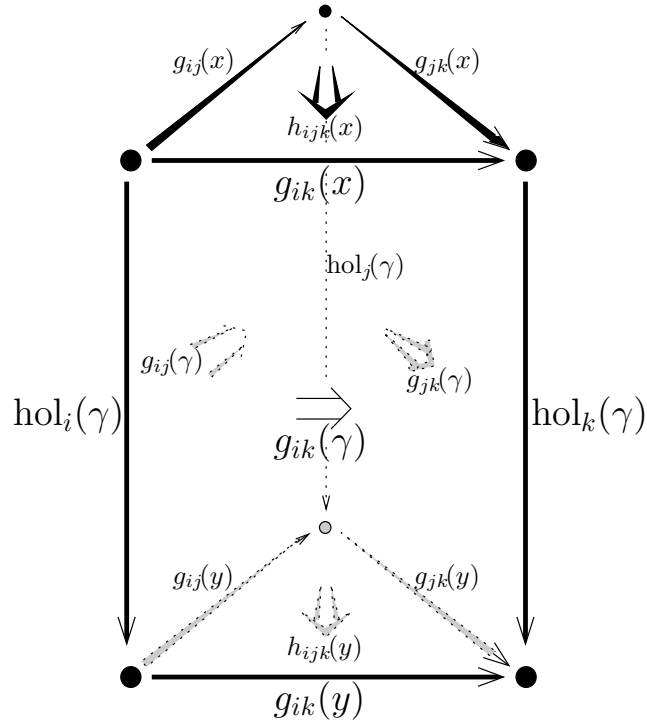


And, we require that:

- for each  $i, j, k$  the function  $h_{ijk}$  defines a modification:



In other words, this diagrams commutes for any path  $\gamma: x \rightarrow y$  in  $U_i \cap U_j \cap U_k$ :



**Theorem.** Suppose that  $E \rightarrow M$  is a  $\mathcal{G}$ -2-bundle with local trivializations over open sets  $U_i$  covering  $M$ . Then there is a one-to-one correspondence between 2-connections on  $E$  and Lie-algebra-valued differential forms  $(A_i, B_i, a_{ij})$  satisfying certain equations:

- The holonomy 2-functor  $\text{hol}_i$  is specified by an  $\mathfrak{g}$ -valued 1-form  $A_i$  and an  $\mathfrak{h}$ -valued 2-form  $B_i$  on  $U_i$ , satisfying the fake flatness condition:

$$dA_i + A_i \wedge A_i + dt(B_i) = 0$$

- The pseudonatural isomorphism  $\text{hol}_i \xrightarrow{g_{ij}} \text{hol}_j$  is specified by the transition functions  $g_{ij}$  together with  $\mathfrak{h}$ -valued 1-forms  $a_{ij}$  on  $U_i \cap U_j$ , satisfying the equations:

$$A_i = g_{ij} A_j g_{ij}^{-1} + g_{ij} dg_{ij}^{-1} - dt(a_{ij})$$

$$B_i = \rho(g_{ij})(B_j) + da_{ij} + a_{ij} \wedge a_{ij} + d\rho(A_i) \wedge a_{ij}$$

- For  $g_{ij} \circ g_{jk} \xrightarrow{h_{ijk}} g_{ik}$  to be a modification, the functions  $h_{ijk}$  must satisfy the equation:

$$a_{ij} + \rho(g_{ij})a_{jk} = h_{ijk} a_{ik} h_{ijk}^{-1} + h_{ijk} d\rho(A_i) h_{ijk}^{-1} + h_{ijk} dh_{ijk}^{-1}$$

**Punchline.** Except for fake flatness, these weird-looking equations show up already in Breen and Messing's definition of a *connection on a nonabelian gerbe!* So, 2-bundles and nonabelian gerbes give closely related approaches to higher gauge theory.