Diversity, Entropy and Thermodynamics

John Baez
http://math.ucr.edu/home/baez/biodiversity/

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The Mathematics of Biodiversity
CRM

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Can we connect biodiversity more deeply to thermodynamics?

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- energy, E
- the partition function, Z
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So, all these and much more are available to biodiversity studies.

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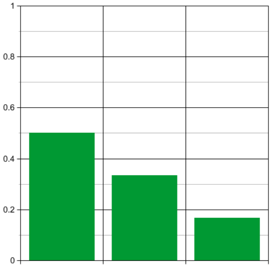
This lets us define probabilities depending on the **temperature** T:

$$p_i(T) = \frac{1}{Z(T)} e^{-E_i/T}$$

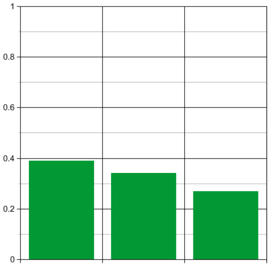
where $Z(T)$ is called the **partition function**:

$$Z(T) = \sum_i e^{-E_i/T}$$

As we raise the temperature, the probabilities $p_i(T)$ become more evenly distributed:



$T = 1$



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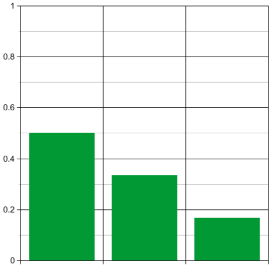

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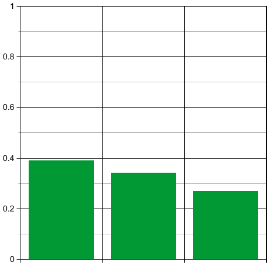

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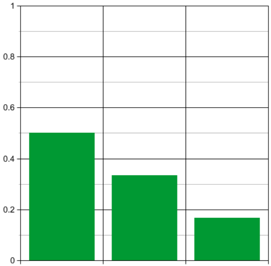
When something gets hotter, all possible situations become closer to being equally probable.

As we lower the temperature, the biggest probabilities increase, while the rest go to zero:



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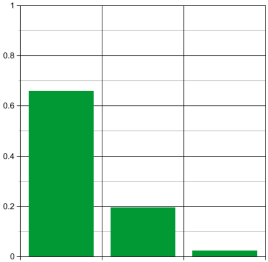

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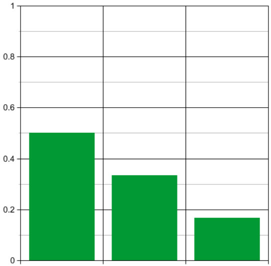

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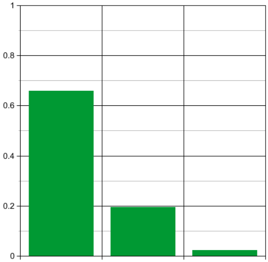

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When something gets colder, the chance that it's in a low-energy state goes up.

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It minimizes the **free energy** F, which we can define for any probability distribution r_i:

$$\begin{aligned} \text{free energy} \quad &= \quad \text{expected energy} \quad - \quad \text{temperature} \quad \text{times} \quad \text{entropy} \\ &= \quad \sum_i r_i E_i \quad + \quad T \sum_i r_i \ln(r_i) \end{aligned}$$

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So:

- When it gets hotter, $p_i(T)$ tries harder to maximize entropy.
- When it gets colder, $p_i(T)$ tries harder to minimize energy.

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Thus, so does the free energy:

$$F(T) = E(T) - TS(T)$$

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No. The **Rényi entropy** is:

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If we let q be the ‘cooling factor’:

$$T = T_0 / q$$

this gives

$$S_q(p) = \frac{1}{1-q} \ln \sum_i e^{-E_i / T} = \frac{1}{1-q} \ln Z(T)$$

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Challenge: If Hill numbers are fundamental to biodiversity, while the partition function is fundamental to thermodynamics, why this funny relationship?

Alternatively, we can write

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Taking $T \to T_0$ we recover a famous formula for the Shannon entropy:

$$S(p) = -\left.\frac{dF(T)}{dT}\right|_{T=T_0}$$

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Challenge: What do these facts mean for biodiversity?

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However, Marc Harper has shown that something *similar* to the Second Law holds when a population approaches an ‘evolutionary optimum’!

Suppose the vector of populations $P = (P_1, \ldots, P_n)$ evolve with time according to a generalized Lotka–Volterra equation:

$$\frac{dP_i}{dt} = f_i(P_1, \ldots, P_n)P_i$$

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Let q be a *fixed* probability distribution, and let

$$I(q,p) = \sum_i \ln\left(\frac{q_i}{p_i}\right) q_i$$

be the **relative Shannon information.**

Then

$$\frac{d}{dt} I(q,p) \le 0$$

if q is an **evolutionary optimum**:

$$p \cdot f(P) \le q \cdot f(P)$$

for all P: i.e., the mean fitness of a small sample of ‘invaders’ distributed according to the distribution q exceeds or equals the mean fitness of any population P.

So: the information ’left to learn’ never increases as the population’s distribution evolves toward an evolutionary optimum. Not biodiversity, but *relative biodiversity*, matters here!