Higher-Dimensional Algebra I: Braided Monoidal 2-Categories

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October 2, 1995

Abstract

We begin with a brief sketch of what is known and conjectured concerning braided monoidal 2-categories and their relevance to 4d TQFTs and 2-tangles. Then we give concise definitions of semistrict monoidal 2-categories and braided monoidal 2-categories, and show how these may be unpacked to give long explicit definitions similar to, but not quite the same as, those given by Kapranov and Voevodsky. Finally, we describe how to construct a semistrict braided monoidal 2-category $\mathcal{Z}(\mathcal{C})$ as the 'center' of a semistrict monoidal category $\mathcal{C}$, in a manner analogous to the construction of a braided monoidal category as the center of a monoidal category. As a corollary this yields a strictification theorem for braided monoidal 2-categories.

1 Introduction

This is the first of a series of articles developing the program introduced in the paper 'Higher-Dimensional Algebra and Topological Quantum Field Theory' [1], henceforth referred to as 'HDA'. This program consists of generalizing algebraic concepts from the context of set theory to the context of n-category theory, and using the resulting language to unify topological quantum field theory with traditional algebraic topology. Rather than doing so systematically from the ground up, the papers in this series will instead address specific issues as they become manageable. The present paper treats a concept which appears to be of special interest in 4-dimensional topology and physics: that of a braided monoidal 2-category.

To understand this concept and its role in higher-dimensional algebra, it is useful to recall some ideas described more thoroughly in HDA. Loosely speaking, an n-category is a structure generalizing a category in which there are 0-morphisms or 'objects', 1-morphisms between objects, 2-morphisms between 1-morphisms, and so on up to n-morphisms. Giving a precise and sufficiently general definition of n-categories is, however, a rather subtle matter. So-called 'strict' n-categories can already be defined recursively for all n, using the idea that for any two objects A and B of an n-category, $\hom(A, B)$ should be not a set but an $(n-1)$-category. One can also unpack this recursive definition and obtain a definition in terms of an explicit list of operations for composing j-morphisms and equational laws the operations obey [30].

However, strict n-categories violate the fundamental principle that *"In any category it is unnatural and undesirable to speak about equality of two objects"* [21]. It is all too easy to mistakenly treat two objects of a category as 'equal' when they are merely isomorphic, so it is better to systematically avoid such mistakes by replacing all equations by specified isomorphisms. Of course, an isomorphism satisfies equations of its own, which state that it is invertible, and in a 2-category these equations themselves should be replaced by specified 2-isomorphisms, and so on. This leads to the recursively defined notion of an 'equivalence': a j-morphism that is strictly invertible if $j = n$, but only invertible up to an equivalence if $j < n$. The practical advantages of replacing equations by specified equivalences are already quite apparent in homotopy theory, and they are likely to become increasingly evident in other branches of mathematics and physics, such as topological quantum field theory.

Taking this philosophy seriously, it is clear that one should define a notion of 'weak' n-category by taking the definition of strict n-category and replacing all equational laws between j-morphisms (for $j < n$) by specified equivalences. To serve essentially the same role as the equations they replace, these equivalences should satisfy some 'coherence laws'. However, to follow the weakening principle, these 'laws' should themselves not be equations, in general, but only specified equivalences, and so on: true equational laws are only to be required at the level of n-morphisms. Unfortunately, determining the correct coherence laws is a rather tricky business, so that weak n-categories have been defined so far only for $n \leq 3$. They are usually called bicategories [2] for $n = 2$ and tricategories [17] for $n = 3$. A major challenge for higher-dimensional algebra is to find a good theory of weak n-categories for all n.

In any event, one expects quite generally that in either the strict or the weak context an $(n+1)$-category $\tilde{\mathcal{C}}$ with only one object $*$ can be regarded as an n-category $\mathcal{C}$ by re-indexing, the j-morphisms of $\mathcal{C}$ being simply the $(j+1)$-morphisms of $\tilde{\mathcal{C}}$. The n-categories we obtain this way have extra structure. For example, since the objects of $\mathcal{C}$ are really morphisms in $\tilde{\mathcal{C}}$ from $*$ to itself, we can 'tensor' or compose them. A category equipped with tensor products is known as a monoidal category, and by analogy we call any n-category arising from an $(n+1)$-category with one object in this way a 'monoidal n-category'. Strict monoidal n-categories are well understood for all n, while the weak ones are presently defined only for $n \leq 2$, since weak n-categories are only defined for $n \leq 3$.

Similarly, we expect that an $(n+2)$-category $\tilde{\mathcal{C}}$ with only one object $*$ and one 1-morphism 1_* can be regarded as an n-category $\mathcal{C}$ with still further structure. In particular, the tensor product should satisfy a kind of commutativity condition. When $n = 0$, this commutativity condition is simply the equation $x \otimes y = y \otimes x$, and it follows from a beautiful argument used by Eckmann and Hilton [12] to show the commutativity of the higher homotopy groups. For simplicity, let $\mathcal{C}$ be a strict 2-category with only one object $*$ and one 1-morphism 1_*. Then for any 2-morphisms x and y in $\tilde{\mathcal{C}}$, both the horizontal composite $x \otimes y$ and the vertical composite xy are well-defined. The 2-morphism $1 = 1_{1_*}$ is the unit for both vertical and horizontal composition, and the exchange identity

$$(x_1 \otimes x_2)(y_1 \otimes y_2) = (x_1 y_1) \otimes (x_2 y_2)$$

holds for all 2-morphisms x_i, y_i. Thus we have

$$\begin{aligned} x \otimes y &= (x1) \otimes (1y) \\ &= (x \otimes 1)(1 \otimes y) \\ &= xy \\ &= (1 \otimes x)(y \otimes 1) \\ &= (1y) \otimes (x1) \\ &= y \otimes x, \end{aligned}$$

so vertical and horizontal composition are equal and $\mathcal{C}$ is a commutative monoid. Conversely, any commutative monoid is the set of 2-morphisms in some 2-category with one object and one 1-morphism.

As a consequence of the philosophy underlying weak n-categories, when $n = 1$ this commutativity condition is not an equation but an isomorphism. In other words, a weak 3-category with only one object and one 1-morphism can be thought of as a weak 'braided' monoidal category: one equipped with a natural isomorphism

$$R_{x,y}: x \otimes y \to y \otimes x$$

satisfying certain coherence laws [17, 19]. More generally, we may define a 'braided monoidal n-category' to be an $(n+2)$-category with one object and one 1-morphism. More generally still, a $(n+k)$-category with only one j-morphism for each $j < k$ can be regarded as a special sort of n-category, a 'k-tuply monoidal n-category'. These play a key role in HDA, from which the table in Figure 1 is taken. Note in particular the 'stabilization' predicted for $k \geq n+2$.

	$n = 0$	$n = 1$	$n = 2$
$k = 0$	sets	categories	2-categories
$k = 1$	monoids	monoidal categories	monoidal 2-categories
$k = 2$	commutative monoids	braided monoidal categories	braided monoidal 2-categories
$k = 3$	' '	symmetric monoidal categories	weakly involutory monoidal 2-categories
$k = 4$	' '	' '	strongly involutory monoidal 2-categories
$k = 5$	' '	' '	' '

1. Weak k-tuply monoidal n-categories: expected results

Unfortunately, the weak versions of these structures have only been defined in certain cases so far. In particular, the weak version of braided monoidal 2-categories is not yet understood, because they should be weak 4-categories with only one object and one 1-morphism, and weak 4-categories have not yet been defined. However, Kapranov and Voevodsky [21] have defined a more limited class of 'semistrict' braided monoidal 2-categories, the hope being that eventually all weak braided monoidal 2-categories could be proven equivalent to these semistrict ones (in some appropriate sense). This strategy has already proven successful at other levels. For example, Gordon, Power, and Street [17] showed that all weak 3-categories are equivalent to a certain class of semistrict ones, and as a corollary, all weak monoidal 2-categories are equivalent to certain semistrict ones. Since braided monoidal 2-categories can be thought of as monoidal 2-categories equipped with extra structure, one expects a similar 'strictification theorem' to hold at the level of braided monoidal 2-categories.

Kapranov and Voevodsky's definition of a semistrict braided monoidal 2-category consists of a long explicit list of operations and equational laws. The first main goal of this paper is to present a more concise and conceptual definition. When we unpack this definition to obtain an explicit list of operations and laws, we find that it differs from Kapranov and Voevodsky's list in a few places. These appear to be slight defects in their definition; for example, our subsequent theorems would not work as smoothly if we used their definition.

1.1 The Center Construction

The second main goal of this paper is to give a procedure for constructing a braided monoidal 2-category as the 'center' $\mathcal{Z}(\mathcal{C})$ of of a monoidal 2-category $\mathcal{C}$. To appreciate this rather complicated procedure it is necessary to understand the general concept of 'center' proposed in HDA. In essence this concept is simple; all the complications arise from the lack of a good general theory of weak n-categories.

There is no 'set of all sets', but there is a class of all sets. Better still, there is a category Set having sets as objects and functions between them as morphisms. Similarly, there is a 2-category Cat having small categories as objects, functors between them as 1-morphisms, and natural transformations between functors as 2-morphisms. In general, we expect there to be a very important $(n+1)$-category nCat having as objects all small n-categories (i.e., those for which the j-morphisms form a set). This has been worked out quite generally in the strict context, but in the weak context only for $n \leq 2$ [2, 17].

In terms of this idea, the 'center' of a small k-tuply monoidal n-category $\mathcal{C}$ is a small $(k+1)$-tuply monoidal n-category $\mathcal{Z}(\mathcal{C})$ defined as follows. Recall that $\mathcal{C}$ is really a special sort of $(n+k)$-category, namely one with only one j-morphism for $j < k$.

Thus $\mathcal{C}$ is an object in $(n+k)$Cat. Let $1_1 = 1_{\mathcal{C}}$ denote the identity 1-morphism of $\mathcal{C}$ in $(n+k)$Cat, and recursively define

$$1_{j+1} = 1_{1_j},$$

so that 1_j is a j-morphism. Then there should be a sub-$(n+k)$-category $\mathcal{Z}(\mathcal{C})$ of $(n+k)$Cat having $\mathcal{C}$ as its only object, $1_{\mathcal{C}}$ as its only 1-morphism, $1_{1_{\mathcal{C}}}$ as its only 2-morphism, and so on up to 1_k, and then having all $(k+1)$-morphisms from 1_k to itself as $(k+1)$-morphisms, all $(k+2)$-morphisms between these as $(k+2)$-morphisms, and so on. Since $\mathcal{Z}(\mathcal{C})$ has only one j-morphism for $j < k+1$, it follows that $\mathcal{Z}(\mathcal{C})$ is a $(k+1)$-tuply monoidal n-category.

As this construction is a bit mind-boggling at first sight, let us illustrate it in the case $n = 0$, $k = 1$. Thus we begin with a small category $\mathcal{C}$ with only one object $*$. The set $\tilde{\mathcal{C}}$ of 1-morphisms of $\mathcal{C}$ can be an arbitrary monoid. Similarly, $\mathcal{Z}(\mathcal{C})$ is a 2-category with only one object and one 1-morphism, and the 2-morphisms of such a 2-category form a commutative monoid. More precisely, $\mathcal{Z}(\mathcal{C})$ is the sub-2-category of Cat having $\mathcal{C}$ as its only object, $1_{\mathcal{C}}$ as its only 1-morphism, and all natural transformations $T\colon 1_{\mathcal{C}} \to 1_{\mathcal{C}}$ as 2-morphisms. What is such a natural transformation in concrete terms? It must assign to the one object $*$ of $\mathcal{C}$ a morphism $T_*\colon * \to *$, such that for all $f\colon * \to *$ the following diagram commutes:

$$\begin{tikzcd} * \arrow[r, "f"] \arrow[d, "T_*"'] & * \arrow[d, "T_*"] \\ * \arrow[r, "f"'] & * \end{tikzcd}$$

In other words, it is simply an element T_* of the center of $\tilde{\mathcal{C}}$. Thus the generalized concept of center reduces in this case to the standard notion.

The case $n = 1$, $k = 1$ is more interesting. The center of a weak monoidal category is a weak braided monoidal category [19, 21, 26]. In particular, if H is a Hopf algebra, the category Reps(H) of finite-dimensional comodules of H is a weak monoidal category, and the center $\mathcal{Z}(\text{Reps}(H))$ is then the category of representations of a coquasitriangular Hopf algebra DH called the 'quantum double' of H. (Working with comodules and coquasitriangular Hopf algebras, rather than modules and quasitriangular Hopf algebras, serves as a technical convenience.) The quantum double construction, invented by Drinfeld [11], gives to many interesting coquasitriangular Hopf algebras. In particular, the quantum groups arising from semisimple Lie groups, while not quantum doubles themselves, are straightforward quotients thereof [20]. Thus the center construction can be regarded as an elegant approach to quantum groups, which, as we shall see, makes their appearance in 3-dimensional topology much less mysterious.

The class of theorems known as 'Tannaka-Krein reconstruction theorems' [9, 27, 32] further clarifies the relation between the center construction and quantum doubles. Given a Hopf algebra H, the category $\mathrm{Reps}(H)$ is a $\mathbb{C}$-linear abelian rigid monoidal category and equipped with a faithful $\mathbb{C}$-linear exact monoidal functor to Vect. Conversely, given any such category $\mathcal{C}$ equipped with such a functor to Vect, $\mathcal{C}$ is equivalent to $\mathrm{Reps}(H)$ for some Hopf algebra H unique up to natural isomorphism. A similar theorem holds for H coquasitriangular and $\mathcal{C}$ braided. Thus we may construct the quantum double of H by first forming $\mathrm{Reps}(H)$, then taking the center $Z(\mathrm{Reps}(H))$ of this category, and then applying Tannaka-Krein reconstruction to obtain DH.

It is natural to hope that other cases of the center construction will give interesting analogs of these results. The most interesting case that can be handled with our present limited understanding of weak n-categories is the case $n = 2$, $k = 1$: if $\mathcal{C}$ is a monoidal 2-category, one expects that $\mathcal{Z}(\mathcal{C})$ will be a braided monoidal 2-category. The difficulty with proving this result is that we lack a general theory of weak 4-categories. Thus we do not know the definition of a weak braided monoidal 2-category, and cannot use the expected result that 3Cat forms a weak 4-category. Instead, we need to start with a semistrict monoidal category $\mathcal{C}$, explicitly describe the objects, morphisms, and 2-morphisms of $\mathcal{Z}(\mathcal{C})$, and then rather laboriously prove that it is indeed a semistrict monoidal 2-category.

In fact, it is natural to conjecture a kind of 'categorification' of the whole theory of quantum doubles. For example, one should be able to start with a 'Hopf category' as defined by Crane and Frenkel [7] — or, better, a 'Hopf 2-algebra' — and form the monoidal 2-category $\mathrm{Reps}(H)$ of its representations on '2-vector spaces' [21, 34]. The monoidal 2-category $\mathrm{Reps}(H)$ should be equipped with a monoidal 2-functor to 2Vect and satisfy various other conditions, and there should be a Tannaka-Krein theorem saying that, conversely, such data determine a Hopf 2-algebra, unique up to equivalence. The center $\mathcal{Z}(\mathrm{Reps}(H))$ should thus be a braided monoidal 2-category, and by Tannaka-Krein reconstruction should determine a Hopf 2-algebra DH, the 'quantum double' of H. Finally, one expects that this quantum double will be 'quasitriangular' in the sense defined by Crane and Frenkel [7]. More ambitiously, one might conjecture a similar correspondence between braided monoidal n-categories and quasitriangular Hopf n-

algebras for higher n. We shall not attempt to make these conjectures precise and prove them here. However, it is helpful to keep them in mind when considering the applications of braided monoidal 2-categories to topology.

1.2 Applications to 4-Dimensional TQFT

Braided monoidal categories are especially interesting because they give efficient procedures for constructing tangle invariants and 3-dimensional topological quantum field theories (TQFTs). Braided monoidal 2-categories appear to have analogous applications to 2-tangle invariants and 4-dimensional TQFTs. As the TQFT applications are more intimately related to the center construction, we begin with these. To see the patterns involved, it is helpful to consider first the rather trivial case of 2-dimensional TQFTs.

A 2-dimensional TQFT is a particular sort of symmetric monoidal functor $\mathcal{F}\colon 2\mathrm{Cob} \to \mathrm{Vect}$. Here the category 2Cob has compact oriented 1-manifolds as objects and compact oriented cobordisms between them as morphisms, and it has a monoidal structure given by disjoint union. Similarly, the category Vect of finite-dimensional vector spaces and linear maps has a monoidal structure given by the usual tensor product. In both cases these categories have a natural symmetric structure, as described in HDA and the references therein. The sphere with 3 open discs removed, or 'trinion', can be thought of as a morphism in 2Cob:

$$m\colon S^1 \cup S^1 \to S^1,$$

and it gives rise to a product on the vector space $\mathcal{F}(S^1)$:

$$\mathcal{F}(m)\colon \mathcal{F}(S^1) \otimes \mathcal{F}(S^1) \to \mathcal{F}(S^1).$$

One can easily check that this product is associative and commutative. Similarly, the closed disc can be thought of as a morphism

$$i\colon \emptyset \to S^1,$$

which gives rise to a unit for the product on $\mathcal{F}(S^1)$:

$$\mathcal{F}(i)\colon \mathbb{C} \to \mathcal{F}(S^1).$$

Thus any 2-dimensional TQFT assigns to the circle a commutative algebra.

The true significance of this fact takes a bit of work to unearth. First, we can define a 'commutative monoid object' in any symmetric monoidal category to be an object A equipped with a product and unit

$$m: A \otimes A \to A, \qquad i: 1 \to A$$

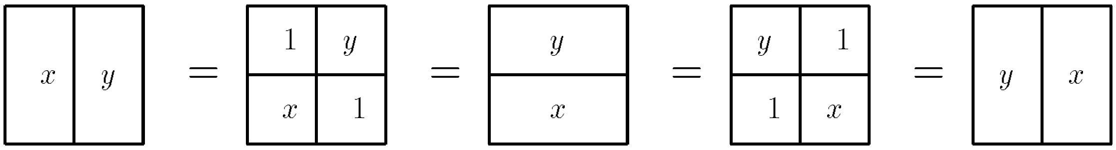
satisfying analogs of the axioms for a commutative monoid. In particular, $\mathcal{F}(S^1)$ is a commutative monoid object in Vect, that is, a commutative algebra. However, this is really just a corollary of the fact that S^1 is a commutative monoid object in 2Cob, since a symmetric monoidal functor takes commutative monoid objects to commutative monoid objects. The real question is therefore, *why is S^1 a commutative monoid object in* 2Cob*?*

We shall not address this question directly. Instead, note that whenever A is a commutative monoid object in a symmetric monoidal category, $\hom(1, A)$ is a commutative monoid. Thus $\hom(\emptyset, S^1)$ is a commutative monoid. Conversely, understanding this commutative monoid should help us understand why S^1 is a commutative monoid object. Moreover, by following the patterns in Figure 1, we can learn something about the role of braided monoidal categories for 3-dimensional TQFTs, and braided monoidal 2-categories in 4-dimensional TQFTs.

An element of $\hom(\emptyset, S^1)$ is an equivalence class of compact oriented 2-manifolds M whose boundary has been identified with S^1. Alternatively, by fitting the circle inside a square in a standard way, we can think of M as a 2-manifold with corners whose boundary is a square. Then, given $x, y \in \hom(\emptyset, S^1)$ we can define a 'vertical' product xy and a 'horizontal' product $x \otimes y$ as shown in Figure 2.

2. Vertical and horizontal product in $\hom(\emptyset, S^1)$

These products satisfy the exchange identity, and taking M to be the disc gives an element $1 \in \hom(\emptyset, S^1)$ that is a unit for both the horizontal and vertical product. The Eckmann-Hilton argument then implies that $\hom(\emptyset, S^1)$ is a commutative monoid. We depict this argument graphically in Figure 3.



3. The Eckmann-Hilton argument

The appearance of the Eckmann-Hilton argument here suggests that we really have a 2-category with one object and one 1-morphism on our hands. Now, the 'extended TQFT hypothesis' in HDA suggests that the best way to understand n-dimensional TQFTs is in terms of a weak n-category $\mathcal{C}_{n,\infty}$ whose objects are 0-manifolds, whose morphisms are equivalence classes of 1-manifolds with boundary, whose 2-morphisms are equivalence classes of 2-manifolds with corners, and so on, each $(j+1)$-morphism being a kind of cobordism between j-morphisms. (Of course these manifolds should be compact and oriented; in general they should also be 'framed', but here we neglect this subtlety.) Making this hypothesis precise would require a general definition of weak n-categories, and also some careful differential topology. Even in its current vague form, though, it sheds some light on the situation at hand. $\mathcal{C}_{n,\infty}$ should have a distinguished object $*$, the positively oriented point. The 1-morphism 1_* should then correspond to the closed unit interval. When $n = 2$, $\hom(1_*, 1_*)$ should then be the set of all cobordisms from the interval to itself. These are just equivalence classes of 2-manifolds with corners whose boundary is the square! Thus $\hom(1_*, 1_*)$ is isomorphic to $\hom(\emptyset, S^1)$, but now the commutative monoid structure has a purely algebraic explanation: there is a 2-category having one object $*$, one 1-morphism 1_*, and the set $\hom(1_*, 1_*)$ as its 2-morphisms. Understand the isomorphism between $\hom(\emptyset, S^1)$ and $\hom(1_*, 1_*)$ in purely algebraic terms remains an interesting challenge; the solution will probably involve the theory of duality in n-categories.

Similarly, in the study of 3-dimensional TQFTs we expect to have a 3-category $\mathcal{C}_{3,\infty}$, and sitting inside this there should be a 3-category with one object $*$, one 1-morphism 1_*, and the category $\hom(1_*, 1_*)$ as its 2-morphisms and 3-morphisms. This category should thus be a braided monoidal category whose objects are 2-manifolds with corners having a square as boundary, and whose morphisms are cobordisms between these. Likewise, in the 4-dimensional case $\hom(1_*, 1_*)$ would be a braided monoidal 2-category, and so on.

In fact, results along these lines already appear in the literature in the cases of dimensions 3 and 4, but in terms of $\hom(\emptyset, S^1)$ rather than $\hom(1_*, 1_*)$. This is less natural algebraically, but simpler topologically, because the theory of cobordisms between manifolds with corners is not well developed. So far, the clearest description of $\hom(\emptyset, S^1)$ as a braided monoidal category in dimension 3 and a braided monoidal 2-category in dimension 4 has been given by Crane and Yetter [8]. There are many interesting projects left to do, however. For example, in dimension 3 it should be possible to use existing results of Kerler [24] and others to obtain a presentation of $\hom(\emptyset, S^1)$ as a braided monoidal category, and to compare the answer to what one would predict using the extended TQFT hypothesis. This presentation should explain the already known conditions required to construct 3-dimensional TQFTs, such as Chern-Simons theory, which associate a braided monoidal category to the circle [6, 29]. In dimension 4 one still needs to carefully check whether $\hom(\emptyset, S^1)$ meets our definition of a braided monoidal 2-category, and then if possible obtain a presentation of it. This may allow the construction of 4-dimensional TQFTs from braided monoidal 2-categories meeting certain conditions. If so, our center construction may serve as a source of 4-dimensional TQFTs.

1.3 Applications to 2-Tangles

Tangles can be regarded as certain equivalence classes of 1-manifolds with boundary embedded in $[0,1]^3$, possibly equipped with extra structure such as an orientation or framing. Tangles are important because they make clear the relation between knot theory and braided monoidal categories. For example, framed oriented tangles form the 'free balanced braided monoidal category on one object' [16, 19, 31, 33], and this fact permits the construction of knot invariants from the categories of representations of quantum groups and other quasitriangular Hopf algebras [28].

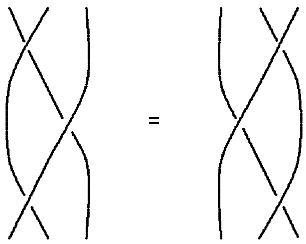
The 'tangle hypothesis' of HDA suggests that this is part of a more general relationship between 'k-tangles in $(n+k)$ dimensions' and k-tuply monoidal n-categories. A k-tangle in $(n+k)$ dimensions is something like an isotopy equivalence class of k-manifolds with corners embedded in $[0,1]^{n+k}$. The tangle hypothesis proposes that these may be described algebraically using a specific k-tuply monoidal n-category $\mathcal{C}_{n,k}$, which has the cobordism n-category $\mathcal{C}_{n,\infty}$ as a limiting case.

A very interesting example is the case of 2-tangles in 4 dimensions: $n=2$, $k=2$. Topologists have already studied these 2-tangles, and the work of Carter and Saito [5] strongly suggests that they form a braided monoidal 2-category. In fact, Fischer [13] claims to have already shown this. His work is unfortunately rather unclear, but Kharlamov and Turaev [25] have begun to redo it more carefully. It should also be re-evaluated in the light of our definition of braided monoidal 2-category. One would eventually like to construct 2-tangle invariants from certain braided monoidal 2-categories, such as the category of representations of quasitriangular Hopf 2-algebras. Our center construction is a small step in this direction.

We cannot conclude this introduction without a word or two about the Zamolodchikov tetrahedron equation. In a braided monoidal category, the braiding automatically satisfies the Yang-Baxter equation. In other words, given objects A, B, C, the following diagram commutes:

$$\begin{tikzcd}
& B \otimes A \otimes C \arrow[r, "B \otimes R_{A,C}"] & B \otimes C \otimes A \arrow[dr, "R_{B,C} \otimes A"] & \\
A \otimes B \otimes C \arrow[ur, "R_{A,B} \otimes C"] \arrow[dr, "A \otimes R_{B,C}"'] & & & C \otimes B \otimes A \\
& A \otimes C \otimes B \arrow[r, "R_{A,C} \otimes B"'] & C \otimes A \otimes B \arrow[ur, "C \otimes R_{A,B}"'] &
\end{tikzcd} \tag{1}$$

In the theory of tangles this corresponds to the following equation between tangles:



4. The Yang-Baxter equation

In a braided monoidal 2-category, the Yang-Baxter equation holds only up to a 2-isomorphism. Topologically, this 2-isomorphism corresponds to a 2-tangle which intersected with $\{0\} \times [0,1]^3 \subset [0,1]^4$ looks like the left side of Figure 4, and which intersected with $\{1\} \times [0,1]^3$ looks like the right side of Figure 4.

In Kapranov and Voevodsky's theory [21] there are in fact two distinct such 2-isomorphisms, $S^{\pm}_{A,B,C}$, corresponding to two distinct *proofs* of the Yang-Baxter equation in a braided monoidal category. However, they give the same 2-tangle. There is also a deep relationship between n-category theory and homotopy theory, described in HDA and the references therein, and using this, Breen [4] has deduced that the condition $S^+ = S^-$ should hold.

These facts constitute topological evidence that in the correct definition of a braided monoidal category, there should be an extra coherence law asserting that $S^+ = S^-$. We also find algebraic evidence for this, as follows. It follows heuristically from our rough definition of center that a k-tuply monoidal n-category should embed canonically in its center when $\mathcal{C}$ happens to be already $(k+1)$-tuply monoidal. More precisely, if $\mathcal{C}$ is a $(k+1)$-tuply monoidal n-category and $\mathcal{C}_0$ is the underlying k-tuply monoidal n-category, there should be a faithful $(k+1)$-tuply monoidal n-functor from $\mathcal{C}$ to $\mathcal{Z}(\mathcal{C}_0)$. For example, the center of a set S works out to be the monoid $\mathrm{End}(S)$, but when S happens already to be a monoid, there is a natural embedding $S \hookrightarrow \mathrm{End}(S)$ given by the left action of S on itself. Similarly, a monoid equals its center when it is commutative, and a monoidal category naturally embeds in its center when it is braided [19, 20]. The third main goal of this paper is to show that a monoidal 2-category $\mathcal{C}$ embeds into $\mathcal{Z}(\mathcal{C})$ when $\mathcal{C}$ happens to be braided. However, for any monoidal 2-category $\mathcal{C}$ it turns out that $S^+ = S^-$ in $\mathcal{Z}(\mathcal{C})$. Thus we can only achieve our goal if our definition of braided monoidal 2-category includes a coherence law saying that $S^+ = S^-$. (It is worth noting that all our results except Theorem 18 hold without this extra coherence law.)

Finally, if $S = S^+ = S^-$, Kapranov and Voevodky's work [21] implies that the 2-morphisms $S_{A,B,C}$ satisfy an equation of their own, the Zamolodchikov tetrahedron equation. This is the higher-dimensional analogue of the Yang-Baxter equation, and it plays an important role in the theory of 2-tangles. Pictures of the Zamolodchikov tetrahedron equation in terms of 2-tangles can be found in the work of Carter and Saito [5]. Kapranov and Voevodsky, who do not assume $S^+ = S^-$, write down 8 different versions of the Zamolodchikov equation and claim that these all follow from their definition of a braided monoidal 2-category. In our framework there is only one Zamolodchikov equation.

2 Definitions

We begin by defining semistrict monoidal 2-categories and semistrict braided monoidal 2-categories. Following traditional practice among category theorists [18, 23], we use '2-category' to mean what Kapranov and Voevodsky [21] call a strict 2-category, and '2-functor' to mean what Kapranov and Voevodsky call a strict 2-functor. Composition of 1-morphisms, the horizontal composition of a 1-morphism and a 2-morphism (in either order) and the horizontal composition of 2-morphisms is denoted by $\circ$ or simply juxtaposition. Vertical composition of 2-morphisms is denoted by $\cdot$. We use the ordering in which, for example, the composite of $f\colon A \to B$ and $g\colon B \to C$ is denoted $f \circ g$.

We use $\mathcal{C} \otimes_{\mathrm{G}} \mathcal{D}$ to denote Gordon, Power and Street's [17] 'Gray' tensor product of the 2-categories $\mathcal{C}$ and $\mathcal{D}$. This differs from Gray's original version [18] in being the 'pseudo' rather than the 'lax' weakening of the Cartesian product. For readers unfamiliar with these distinctions, let us simply recall that given a 1-morphism $f\colon A \to A'$ in $\mathcal{C}$ and a 1-morphism $g\colon B \to B'$ in $\mathcal{D}$, the Cartesian product $\mathcal{C} \times \mathcal{D}$ contains a commuting square

$$\begin{tikzcd}
(A,B) \arrow[r, "f \times 1"] \arrow[d, "1 \times g"'] & (A',B) \arrow[d, "1 \times g"] \\
(A,B') \arrow[r, "f \times 1"'] & (A',B')
\end{tikzcd}$$

Following the 'lax' approach to weakening, which consists of replacing equations by morphisms, Gray's original product of $\mathcal{C}$ and $\mathcal{D}$ instead contains a square commuting only up to a specified 2-morphism:

$$\begin{tikzcd}
(A,B) \arrow[r, "f \otimes_{\mathrm{G}} 1"] \arrow[d, "1 \otimes_{\mathrm{G}} g"'] \arrow[dr, phantom, "\Downarrow \gamma_{f,g}"] & (A',B) \arrow[d, "1 \otimes_{\mathrm{G}} g"] \\
(A,B') \arrow[r, "f \otimes_{\mathrm{G}} 1"'] & (A',B')
\end{tikzcd}$$

Following the 'pseudo' approach, which consists of replacing equations by isomorphisms (or, more generally, equivalences), Gordon, Power, and Street additionally require $\gamma_{f,g}$ to be an isomorphism. We use their version of the Gray tensor product as part of a systematic adherence to the 'pseudo' approach.

The category 2Cat with 2-categories as objects and 2-functors as morphisms becomes a monoidal category $(2\mathrm{Cat}, \otimes_{\mathrm{G}}, \mathcal{I})$ when equipped with the Gray tensor product and the unit object $\mathcal{I}$, the 2-category with one object, one morphism and one 2-morphism. This monoidal category is symmetric, with the symmetry

$$S_{\mathcal{C},\mathcal{D}}\colon \mathcal{C} \otimes_{\mathrm{G}} \mathcal{D} \to \mathcal{D} \otimes_{\mathrm{G}} \mathcal{C}$$

given by:

$$\begin{array}{ccc}
(B,A) \mapsto (B,A) & (f,1) \mapsto (1,f) & (1,g) \mapsto (g,1) \\
(\alpha,1) \mapsto (1,\alpha) & (1,\beta) \mapsto (\beta,1) & \gamma_{f,g} \mapsto \gamma_{g,f}^{-1}
\end{array}$$

2.1 Semistrict Monoidal 2-Categories

Since 2Cat is monoidal when equipped with the Gray tensor product, we may use enriched category theory [22] to efficiently define semistrict 3-categories and monoidal 2-categories:

Definition 1. *A* semistrict 3-category *is a category enriched over* $(2\mathrm{Cat}, \otimes_{\mathrm{G}}, I)$.

Definition 2. *A* semistrict monoidal 2-category *is a semistrict 3-category with one object.*

Gordon, Power and Street [17] have given a definition of 'weak' 3-categories, or 'tricategories', seemingly more general than that of semistrict 3-categories, and indeed intended to be 'maximally general' in some sense. For example, associativity and identity laws hold as equations in a semistrict 3-category, but only hold up to specified equivalence in a weak one. However, these authors have shown that every weak 3-category is equivalent in a precise sense ('triequivalence') to a semistrict one, so for many purposes semistrict 3-categories are 'sufficiently general'. Defining a weak monoidal 2-category to be a weak 3-category with one object, it follows from their proof that any one of these is triequivalent to a semistrict monoidal 2-category. So again, while not maximally general, semistrict monoidal 2-categories are sufficiently general for many purposes.

Often we shall think of a semistrict monoidal 2-category as a 2-category with extra structure. More precisely, if $\tilde{\mathcal{C}}$ is a semistrict 3-category with one object $*$, let $\mathcal{C} = \hom(*, *)$. This is a 2-category equipped with a 2-functor

$$\otimes : \mathcal{C} \otimes_{\mathrm{G}} \mathcal{C} \to \mathcal{C}$$

coming from composition in $\tilde{\mathcal{C}}$, as well as a functor $i \colon \mathcal{I} \to \mathcal{C}$ coming from the identity of $*$ in $\tilde{\mathcal{C}}$.

Lemma 3. *Suppose $\tilde{\mathcal{C}}$ is a semistrict 3-category with one object, and let $(\mathcal{C}, \otimes, i)$ be defined as above. Then $\mathcal{C}$ is a 2-category, $\otimes \colon \mathcal{C} \otimes_{\mathrm{G}} \mathcal{C} \to \mathcal{C}$ and $i \colon \mathcal{I} \to \mathcal{C}$ are 2-functors, and the following diagrams commute:*

1. *Associativity:*

$$\begin{tikzcd}
\mathcal{C} \otimes_{\mathrm{G}} \mathcal{C} \otimes_{\mathrm{G}} \mathcal{C} \arrow[r, "\otimes \otimes_{\mathrm{G}} \mathcal{C}"] \arrow[d, "\mathcal{C} \otimes_{\mathrm{G}} \otimes"'] & \mathcal{C} \otimes_{\mathrm{G}} \mathcal{C} \arrow[d, "\otimes"] \\
\mathcal{C} \otimes_{\mathrm{G}} \mathcal{C} \arrow[r, "\otimes"'] & \mathcal{C}
\end{tikzcd}$$

2. *Unit law:*

$$\begin{tikzcd}
\mathcal{I} \otimes_{\mathrm{G}} \mathcal{C} \arrow[rr, "i \otimes_{\mathrm{G}} \mathcal{C}"] \arrow[dr, "\cong"'] & & \mathcal{C} \otimes_{\mathrm{G}} \mathcal{C} \arrow[dl, "\otimes"] \\
& \mathcal{C} &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C} \otimes_{\mathrm{G}} \mathcal{I} \arrow[rr, "\mathcal{C} \otimes_{\mathrm{G}} i"] \arrow[dr, "\cong"'] & & \mathcal{C} \otimes_{\mathrm{G}} \mathcal{C} \arrow[dl, "\otimes"] \\
& \mathcal{C} &
\end{tikzcd}$$

Conversely, for any $(\mathcal{C}, \otimes, i)$ with these properties, there is a unique semistrict 3-category $\tilde{\mathcal{C}}$ with one object from which $(\mathcal{C}, \otimes, i)$ arises as above.

Proof - This is a straightforward consequence of the definition of semistrict 3-categories as categories enriched over 2Cat with its Gray tensor product. $\square$

There is thus no harm in thinking of a semistrict monoidal 2-category as a triple $(\mathcal{C}, \otimes, i)$ satisfying the associativity and unit law conditions of Lemma 3. Since the 2-functor i is determined by the object of $\mathcal{C}$ obtained by applying $i: \mathcal{I} \to \mathcal{C}$ to the one object in $\mathcal{I}$, we can also think of a semistrict monoidal 2-category as a triple $(\mathcal{C}, \otimes, I)$.

One may further unpack our definition of a semistrict monoidal 2-category and obtain the same explicit list of operations and laws that Kapranov and Voevodsky take as their definition [21]. Here the standard machinery of 2-categorical commutative diagrams becomes very handy [23]. In what follows we write $\otimes_{f,g}$ for the 2-morphism $\otimes(\gamma_{f,g})$ in $\mathcal{C}$.

Lemma 4. *A semistrict monoidal 2-category consists of a 2-category $\mathcal{C}$ together with:*

1. *An object $I \in \mathcal{C}$.*

2. *For any two objects $A, B \in \mathcal{C}$ an object $A \otimes B$ in C.*

3. *For any 1-morphism $f: A \to A'$ and any object $B \in \mathcal{C}$ a 1-morphism $f \otimes B: A \otimes B \to A' \otimes B$.*

4. *For any 1-morphism $g: B \to B'$ and any object $A \in \mathcal{C}$ a 1-morphism $A \otimes g: A \otimes B \to A \otimes B'$.*

5. *For any object $B \in \mathcal{C}$ and any 2-morphism $\alpha: f \Rightarrow f'$ a 2-morphism $\alpha \otimes B: f \otimes B \Rightarrow f' \otimes B$.*

6. *For any object $A \in \mathcal{C}$ and any 2-morphism $\beta: g \Rightarrow g'$ a 2-morphism $A \otimes \beta: A \otimes g \Rightarrow A \otimes g'$.*

7. *For any two 1-morphisms $f: A \to A'$ and $g: B \to B'$ a 2-isomorphism*

$$\begin{tikzcd}
A \otimes B \arrow[r, "A\otimes g"] \arrow[d, "f\otimes B"'] \arrow[dr, phantom, "\Downarrow \otimes_{f,g}"] & A \otimes B' \arrow[d, "f\otimes B'"] \\
A' \otimes B \arrow[r, "A'\otimes g"'] & A' \otimes B'
\end{tikzcd}$$

Moreover, these data must satisfy the following conditions.

(i) *For any object $A \in \mathcal{C}$ we have $A \otimes \text{-}: \mathcal{C} \to \mathcal{C}$ and $\text{-} \otimes A: \mathcal{C} \to \mathcal{C}$ are 2-functors.*

(ii) *For x any object, morphism or 2-morphism of $\mathcal{C}$ we have $x \otimes I = I \otimes x = x$.*

(iii) *For x any object, morphism or 2-morphism of $\mathcal{C}$, and for all objects $A, B \in \mathcal{C}$ we have $A \otimes (B \otimes x) = (A \otimes B) \otimes x$, $A \otimes (x \otimes B) = (A \otimes x) \otimes B$ and $x \otimes (A \otimes B) = (x \otimes A) \otimes B$.*

(iv) *For any 1-morphisms $f: A \to A'$, $g: B \to B'$ and $h: C \to C'$ in $\mathcal{C}$ we have $\bigotimes_{A\otimes g,h} = A \bigotimes \otimes_{g,h}$, $\bigotimes_{fy\otimes B,h} = \bigotimes_{f,B\otimes h}$ and $\bigotimes_{f,g\otimes C} = \bigotimes_{f,g} \otimes C$.*

(v) *For any objects $A, B \in \mathcal{C}$ we have $1_A \otimes B = A \otimes 1_B = 1_{A\otimes B}$, and for any 1-morphisms $f: A \to A'$, $g: B \to B'$ in $\mathcal{C}$ we have $\bigotimes_{1_A,g} = 1_{A\otimes g}$ and $\bigotimes_{f,1_B} = 1_{f\otimes B}$.*

(vi) *For any 1-morphism $f : A \to A'$, any 1-morphisms $g, g': B \to B'$, and any 2-morphism $\beta: g \Rightarrow g'$ the following diagram commutes:*

$$\begin{tikzcd}[column sep=huge, row sep=huge]
A \otimes B \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow A\otimes\beta"] \arrow[d] \arrow[d, phantom, "\Downarrow \otimes_{f,g}"] & A \otimes B' \arrow[d] \arrow[d, phantom, "\Downarrow \otimes_{f,g'}"] \\
A' \otimes B \arrow[r, bend left, dashed] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow A'\otimes\beta"] & A' \otimes B'
\end{tikzcd}$$

(vii) For any 1-morphism $g : B \to B'$, any 1-morphisms $f, f' \colon A \to A'$, and any 2-morphism $\alpha\colon f \Rightarrow f'$, the following diagram commutes:

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
A\otimes B \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow \alpha\otimes B"] \arrow[d] \arrow[dr, phantom, "\Uparrow \otimes_{f',g} \qquad \Uparrow \otimes_{f,g}"] & A'\otimes B \arrow[d] \\
A\otimes B' \arrow[r, bend left, dashed] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow \alpha\otimes B'"] & A'\otimes B'
\end{tikzcd}
$$

(viii) For any 1-morphisms $f\colon A \to A'$, $g\colon B \to B'$ and $g'\colon B' \to B''$ the 2-isomorphism $\bigotimes_{f,gg'}$ coincides with the pasting of $\bigotimes_{f,g}$ and $\bigotimes_{f,g'}$ as in the following diagram.

$$
\begin{tikzcd}
A\otimes B \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow\otimes_{f,g}"] & A\otimes B' \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow\otimes_{f,g'}"] & A\otimes B'' \arrow[d] \\
A'\otimes B \arrow[r] & A'\otimes B' \arrow[r] & A'\otimes B''
\end{tikzcd}
$$

For any 1-morphisms $f\colon A \to A'$, $f'\colon A' \to A''$ and $g\colon B \to B'$ the 2-isomorphism $\bigotimes_{ff',g}$ coincides with the pasting of $\bigotimes_{f,g}$ and $\bigotimes_{f,g'}$ in a similar way.

Proof - This is a straightforward verification. In particular, conditions (v), (vi) and (vii) come from the coherence laws satisfied by $\gamma_{f,g}$ in the Gray tensor product. $\square$

Note that condition $(viii)$ and the invertibility of the 2-morphism $\otimes_{f,g}$ imply that $\otimes_{1_A,g} = 1_g$ and $\otimes_{f,1_B} = 1_f$, for any $f : A \to A'$ and any $g : B \to B'$.

2.2 Semistrict Braided Monoidal 2-Categories

To efficiently define braided monoidal 2-categories it is useful to exploit the fact that $(2\mathrm{Cat}, \otimes_{\mathrm{G}}, \mathcal{I})$ is closed, i.e., enriched over itself [17]. Put more explicitly, what this means is that 2Cat can be regarded as a semistrict 3-category having small 2-categories as objects, 2-functors as morphisms, 'pseudonatural transformations' as 2-morphisms, and 'modifications' as 3-morphisms [3, 23]. A pseudonatural transformation T between 2-functors $\mathcal{F}, \mathcal{G}: \mathcal{C} \to \mathcal{D}$ assigns to each object $A \in \mathcal{C}$ a morphism $T_A: \mathcal{F}(A) \to \mathcal{G}(A)$ which satisfies the definition of a natural transformation only *up to a specified isomorphism.* Thus, T also assigns to each morphism $f: A \to B$ in $\mathcal{C}$ a 2-isomorphism T_f as follows:

$$
\begin{tikzcd}
\mathcal{F}(A) \arrow[r, "\mathcal{F}(f)"] \arrow[d, "T_A"'] \arrow[dr, phantom, "\Downarrow T_f"] & \mathcal{F}(B) \arrow[d, "T_B"] \\
\mathcal{G}(A) \arrow[r, "\mathcal{G}(f)"'] & \mathcal{G}(B)
\end{tikzcd}
$$

These 2-morphisms T_f must in turn satisfy some equational laws of their own. First, for any identity morphism $1_A: A \to A$, we require $T_{1_A} = 1_{T_A}$. Second, given a composable pair of morphisms $f: A \to B$, $g: B \to C$, the 2-morphism T_{fg} is given by the following pasting:

$$
\begin{tikzcd}
\mathcal{F}(A) \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow T_f"] & \mathcal{F}(B) \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow T_g"] & \mathcal{F}(C) \arrow[d] \\
\mathcal{G}(A) \arrow[r] & \mathcal{G}(B) \arrow[r] & \mathcal{G}(C)
\end{tikzcd}
$$

Third, given morphisms $f, f': A \to B$ and a 2-morphism $\alpha: f \Rightarrow f'$, the following diagram commutes:

$$
\begin{tikzcd}
\mathcal{F}(A) \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\mathcal{F}(\alpha)"] \arrow[d] \arrow[dr, phantom, "\Downarrow T_{f'}"] & \mathcal{F}(B) \arrow[d, "\Downarrow T_f"'] \\
\mathcal{G}(A) \arrow[r, bend left, dashed] \arrow[r, bend right] \arrow[r, phantom, "\mathcal{G}(\alpha)"] & \mathcal{G}(B)
\end{tikzcd}
$$

Given two pseudonatural transformations $S, T: \mathcal{F} \Rightarrow \mathcal{G}$, a modification α from S to T assigns to each object $A \in \mathcal{C}$ a 2-morphism $\alpha_A: S_A \Rightarrow T_A$. Moreover, for any morphism $F: A \to B$, the following diagram is required to commute:

$$
\begin{tikzcd}
\mathcal{F}(A) \arrow[rr, bend left] \arrow[rr, bend right] \arrow[rr, phantom, "\Downarrow \alpha_A"] \arrow[d, "\Uparrow T_f"] & & \mathcal{G}(A) \arrow[d, "\Uparrow S_f"'] \\
\mathcal{F}(B) \arrow[rr, bend left, dashed] \arrow[rr, bend right] \arrow[rr, phantom, "\Downarrow \alpha_B"] & & \mathcal{G}(B)
\end{tikzcd}
$$

As explained in the introduction, in an n-category the notion of 'isomorphism' can be weakened to a recursively defined notion of 'equivalence'. In the case of 2Cat this gives the following concepts. A modification α from the pseudonatural transformation S to the pseudonatural transformation T is 'invertible' if there is a modification α^{-1} from T to S such that $\alpha\alpha^{-1} = 1_S$ and $\alpha^{-1}\alpha = 1_T$. A pseudonatural transformation T from $\mathcal{F}$ to $\mathcal{G}$ is a 'pseudonatural equivalence' if there is a pseudonatural transformation $\overline{T}: \mathcal{G} \to F$ and invertible modifications

$$\alpha_1: T\overline{T} \to 1_{\mathcal{F}}, \qquad \alpha_2: \overline{T}T \to 1_{\mathcal{G}}.$$

There is a similar notion at the level of 2-functors, but we will not need it.

Every semistrict monoidal 2-category has a second, 'opposite' tensor product:

Lemma 5 *Suppose $(\mathcal{C}, \otimes, I)$ is a semistrict monoidal 2-category. Then $(\mathcal{C}, \otimes^{\mathrm{op}}, I)$ is also a semistrict monoidal 2-category, where $\otimes^{\mathrm{op}} = S_{\mathcal{C},\mathcal{C}} \circ \otimes$.*

Proof - Straightforward. $\square$

There is an analogous opposite tensor product for strict monoidal categories, and a strict braided monoidal category is just a strict monoidal category equipped with a natural isomorphism $R: \otimes \Rightarrow \otimes^{\mathrm{op}}$, the 'braiding', such that the following triangles commute:

$$
\begin{tikzcd}
A \otimes X \otimes Y \arrow[rr, "R_{A,X \otimes Y}"] \arrow[dr, "R_{A,X} \otimes Y"'] & & X \otimes Y \otimes A \\
& X \otimes A \otimes Y \arrow[ur, "X \otimes R_{A,Y}"'] &
\end{tikzcd}
$$

$$\begin{tikzcd}
X \otimes Y \otimes A \arrow[rr, "R_{X\otimes Y,A}"] \arrow[dr, "X \otimes R_{Y,A}"'] & & A \otimes X \otimes Y \\
& X \otimes A \otimes Y \arrow[ur, "R_{X,A} \otimes Y"'] &
\end{tikzcd}$$

The definition of a semistrict braided monoidal 2-category is very similar. However, instead of a strict monoidal category, one starts with a semistrict monoidal 2-category. Instead of the braiding being a natural transformation, it is a pseudonatural equivalence. Instead of the equations above holding 'on the nose', they hold up to specified invertible modifications. Finally, these modifications must satisfy 3 new coherence laws discovered by Kapranov and Voevodsky, together with the equation $S^+ = S^-$ discussed in Section 1.3.

In all that follows, in diagrams we sometimes denote the tensor product of objects simply by juxtaposition. We also label some clauses in the definition using the 'hieroglyphic' notation invented by Kapranov and Voevodsky.

Definition 6. *A braided monoidal 2-category* $(\mathcal{C}, \otimes, I, R, \tilde{R}_{(-|-,-)}, \tilde{R}_{(-,-|-)})$ *consists of:*

1. *A semistrict monoidal 2-category* $(\mathcal{C}, \otimes, 1)$
2. *A pseudonatural equivalence* $R\colon \otimes \Rightarrow \otimes^{\mathrm{op}}$
3. *Two invertible modifications* $\tilde{R}_{(-|-,-)}$ *and* $\tilde{R}_{(-,-|-)}$*, giving for any objects* $A, B, C \in \mathcal{C}$ *the 2-isomorphisms*

$$\begin{tikzcd}
A \otimes B \otimes C \arrow[rr, "R_{A,B\otimes C}"] \arrow[dr, "R_{A,B} \otimes C"'] \arrow[rr, phantom, shift right=4ex, "\Uparrow \tilde{R}_{(A|B,C)}"] & & B \otimes C \otimes A \\
& B \otimes A \otimes C \arrow[ur, "B \otimes R_{A,C}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
A \otimes B \otimes C \arrow[rr, "R_{A\otimes B,C}"] \arrow[dr, "A \otimes R_{B,C}"'] \arrow[rr, phantom, shift right=4ex, "\Uparrow \tilde{R}_{(A,B|C)}"] & & C \otimes A \otimes B \\
& A \otimes C \otimes B \arrow[ur, "R_{A,C} \otimes B"'] &
\end{tikzcd}$$

These data must satisfy the following conditions. First, for all objects $A, B, C, D \in \mathcal{C}$ the following diagrams commute:

$((\bullet \otimes \bullet \otimes \bullet) \otimes \bullet)$

$$\begin{tikzcd}
 & DABC & \\
ABCD \arrow[ur, "1."] \arrow[rr, dashed, "4."] \arrow[dr] & & ADBC \arrow[ul, "2."'] \\
 & ABDC \arrow[uu] \arrow[ur, "3."'] &
\end{tikzcd}$$

1. $\tilde{R}_{(A\otimes B,C|D)}$ 2. $= \tilde{R}_{(A,B|D)} \otimes C$
3. $= A \otimes \tilde{R}_{(B,C|D)}$ 4. $= \tilde{R}_{(A,B\otimes C|D)}$

$(\bullet \otimes (\bullet \otimes \bullet \otimes \bullet))$

$$\begin{tikzcd}
 & BCDA & \\
ABCD \arrow[ur, "1."] \arrow[rr, dashed, "4."] \arrow[dr] & & BCAD \arrow[ul, "2."'] \\
 & BACD \arrow[uu] \arrow[ur, "3."'] &
\end{tikzcd}$$

1. $= \tilde{R}_{(A|B,C\otimes D)}$ 2. $= B \otimes \tilde{R}_{(A|C,D)}$
3. $= \tilde{R}_{(A|B,C)} \otimes D$ 4. $= \tilde{R}_{(A|B\otimes C,D)}$

$((\bullet \otimes \bullet) \otimes (\bullet \otimes \bullet))$

$$\begin{tikzcd}
ABCD \arrow[rrrr, "5."'] \arrow[drr, dashed, "3."'] \arrow[ddr] \arrow[dddrr, "7."] & & & & CDAB \\
 & & ACDB \arrow[urr, dashed] \arrow[dr, dashed, "4."] & & \\
 & ACBD \arrow[ur, dashed] \arrow[dr, "1."] & \arrow[phantom, "6."] & CADB \arrow[uur, "2."'] & \\
 & & CABD \arrow[uuurr] \arrow[ur] & &
\end{tikzcd}$$

$$
\begin{array}{lll}
1. = \tilde{R}_{(A,B|C)} \otimes D & 2. = C \otimes \tilde{R}_{(A,B|D)} & 3. = A \otimes \tilde{R}_{(B|C,D)} \\
4. = \tilde{R}_{(A|C,D)} \otimes B & 5. = \tilde{R}_{(A,B|C\otimes D)} & 6. = \otimes_{(R_{A,C},R_{B,D})} \\
7. = \tilde{R}_{(A\otimes B|C,D)} & &
\end{array}
$$

Second, for any objects $A, B, C \in \mathcal{C}$, we define two 2-isomorphisms corresponding to two proofs of the Yang-Baxter hexagon in a braided monoidal category:

$$
\begin{tikzcd}
 & BAC \arrow[r] \arrow[d, phantom, "\Downarrow \tilde{R}^{-1}_{(A|B,C)}"] & BCA \arrow[dr] & \\
ABC \arrow[ur] \arrow[urr] \arrow[dr] \arrow[rrr, phantom, "\Downarrow R^{-1}_{(A,R_{B,C})}"] & & & CBA \\
 & ACB \arrow[r] \arrow[urr] & CAB \arrow[ur] \arrow[u, phantom, "\Downarrow \tilde{R}_{(A|B,C)}"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & BAC \arrow[r] \arrow[drr] & BCA \arrow[dr] \arrow[d, phantom, "\Downarrow \tilde{R}_{(A,B|C)}"] & \\
ABC \arrow[ur] \arrow[dr] \arrow[drr] \arrow[rrr, phantom, "\Downarrow R_{(R_{(A,B)},C)}"] & & & CBA \\
 & ACB \arrow[r] \arrow[u, phantom, "\Downarrow \tilde{R}^{-1}_{(A,B|C)}"] & CAB \arrow[ur] &
\end{tikzcd}
$$

We refer to these 2-morphisms as $S^{+}_{A,B,C}$ and $S^{-}_{A,B,C}$, respectively. We require them to be equal: ($S^{+} = S^{-}$):

$$
\begin{tikzcd}
 & BAC \arrow[r] \arrow[drr] & BCA \arrow[dr] & \\
ABC \arrow[ur] \arrow[dr] \arrow[drr] \arrow[urr, dashed] & & & CBA \\
 & ACB \arrow[r] \arrow[urr, dashed] & CAB \arrow[ur] &
\end{tikzcd}
$$

We can unpack this definition to obtain an explicit list of operations and laws. In this form the definition is essentially due to Kapranov and Voevodsky, though with certain differences, which we list at the end of this section.

Lemma 7. *A braided monoidal 2-category $(\mathcal{C}, \otimes, 1, R, \tilde{R}_{(\text{-}|\text{-},\text{-})}, \tilde{R}_{(\text{-},\text{-}|\text{-})})$ consists of the following data:*

1. *A semistrict monoidal 2-category $(\mathcal{C}, \otimes, 1)$*

2. *$(\bullet \otimes \bullet)$ For any two objects $A, B \in \mathcal{C}$ an equivalence $R_{A,B} \colon A \otimes B \to B \otimes A$*

3. *$(\to \otimes \bullet)$ For any 1-morphism $f : A \to A'$ and any object $B \in \mathcal{C}$ a 2-isomorphism*

$$\begin{tikzcd}
A \otimes B \arrow[r, "f\otimes B"] \arrow[d, "R_{A,B}"'] \arrow[dr, phantom, "\Downarrow R_{f,B}"] & A' \otimes B \arrow[d, "R_{A',B}"] \\
B \otimes A \arrow[r, "B\otimes f"'] & B \otimes A'
\end{tikzcd}$$

4. ($\bullet\otimes\to$) *For any object* $A \in \mathcal{C}$ *and any 1-morphism* $g\colon B \to B'$ *a 2-isomorphism*

$$\begin{tikzcd}
A \otimes B \arrow[r, "A\otimes g"] \arrow[d, "R_{A,B}"'] \arrow[dr, phantom, "\Downarrow R_{A,g}"] & A \otimes B' \arrow[d, "R_{A,B'}"] \\
B \otimes A \arrow[r, "g\otimes A"'] & B'\otimes A
\end{tikzcd}$$

5. ($(\bullet\otimes\bullet)\otimes\bullet$) *For any objects* $A, B, C \in \mathcal{C}$ *a 2-iso*

$$\begin{tikzcd}
A\otimes B\otimes C \arrow[rr, "R_{A,B\otimes C}"] \arrow[dr, "R_{A,B}\otimes C"'] & \arrow[d, phantom, "\Uparrow\tilde{R}_{(A|B,C)}"] & B\otimes C\otimes A \\
& B\otimes A\otimes C \arrow[ur, "B\otimes R_{A,C}"'] &
\end{tikzcd}$$

6. ($\bullet\otimes(\bullet\otimes\bullet)$) *For any objects* $A, B, C \in \mathcal{C}$ *a 2-isomorphism*

$$\begin{tikzcd}
A\otimes B\otimes C \arrow[rr, "R_{A\otimes B,C}"] \arrow[dr, "A\otimes R_{B,C}"'] & \arrow[d, phantom, "\Uparrow\tilde{R}_{(A,B|C)}"] & C\otimes A\otimes B \\
& A\otimes C\otimes B \arrow[ur, "R_{A,C}\otimes B"'] &
\end{tikzcd}$$

Moreover, these data must satisfy the following conditions:

($\to\otimes\to$) *For any 1-morphisms* $f\colon A \to A'$ *and* $g\colon B \to B'$ *the following cube commutes:*

$$\begin{tikzcd}
& AB \arrow[rr] \arrow[dl] \arrow[dd, dashed] & 1. & A'B \arrow[dl] \arrow[dd] \\
AB' \arrow[rr] \arrow[dd] & & A'B' \arrow[dd] & 6. \\
3. & BA \arrow[rr, dashed] \arrow[dl, dashed] & 5. & BA' \arrow[dl] \\
B'A \arrow[rr] & 2. & B'A' & 4.
\end{tikzcd}$$

1. $= \otimes_{f,g}$ 2. $= \otimes_{g,f}$ 3. $= R_{A,g}$ 4. $= R_{A',g}$ 5. $= R_{f,B'}$ 6. $= R_{f,B}$

($\bullet\otimes\Downarrow$) *For any object $A \in \mathcal{C}$, any 1-morphisms $f, f'\colon B \to B'$, and any 2-morphism $\beta\colon f \Rightarrow f'$, the following prism commutes:*

$$
\begin{tikzcd}
AB \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow A\otimes\beta"] \arrow[d, "R_{A,f'}"] & AB' \arrow[d] \\
BA \arrow[r, dashed, bend left, "R_{A,f}"] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\beta\otimes A"] & B'A
\end{tikzcd}
$$

($\Downarrow\otimes\bullet$) *A similar prism, left to the reader.*

($\to\to\otimes\bullet$) *For any pair of 1-morphisms $A \xrightarrow{f} A' \xrightarrow{f'} A''$ and any object $B \in \mathcal{C}$, the 2-isomorphism $R_{ff',B}$ coincides with the pasting*

$$
\begin{tikzcd}
A \otimes B \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow R_{f,B}"] & A' \otimes B \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow R_{f',B}"] & A'' \otimes B \arrow[d] \\
B \otimes A \arrow[r] & B \otimes A' \arrow[r] & B \otimes A''
\end{tikzcd}
$$

($\bullet\otimes\to\to$) *A similar pasting law, left to the reader.*

($(\bullet\otimes\bullet)\otimes\to$) *For any objects $A, B, C \in \mathcal{C}$ and any 1-morphism $f\colon C \to C'$, the following triangular prism commutes:*

$$
\begin{tikzcd}
ABC \arrow[rr, "R_{AB,C}"] \arrow[dr] \arrow[dd, "AB\otimes f"'] \arrow[dr, phantom, "\Uparrow 3.", near start] \arrow[ddr, phantom, "\Uparrow 1."] & & CAB \arrow[dd, "f\otimes AB"] \\
& ACB \arrow[ur] \arrow[dd] \arrow[dr, phantom, "\Uparrow 2."] & \\
ABC' \arrow[rr, dashed] \arrow[dr] \arrow[dr, phantom, "\Uparrow 4.", near start] & & C'AB \arrow[l, phantom, "\Uparrow 5."] \\
& AC'B \arrow[ur] &
\end{tikzcd}
$$

1. $= A \otimes R_{B,f}$ 2. $= R_{A,f} \otimes B$ 3. $= \tilde{R}_{(A,B|C)}$ 4. $= \tilde{R}_{(A,B|C')}$ 5. $= R_{AB,f}$

($\to\otimes(\bullet\otimes\bullet)$) *A similar prism, left to the reader.*

($(\to\otimes\bullet)\otimes\bullet$) *For any objects $A, B, C \in \mathcal{C}$ and any 1-morphism $f\colon A \to A'$, the following triangular prism commutes:*

$$
\begin{tikzcd}
ABC \arrow[rr, "R_{AB,C}"] \arrow[dr] \arrow[dd, "f\otimes BC"'] \arrow[r, phantom, shift right=3, "\Uparrow 3."] \arrow[d, phantom, shift left=5, "\Uparrow 1."] & & CAB \arrow[dd, "C\otimes f\otimes B"] \\
& ACB \arrow[ur] \arrow[dd] \arrow[r, phantom, shift right=2, "\Uparrow 2."] & \\
A'BC \arrow[rr, dashed] \arrow[dr] \arrow[r, phantom, shift right=3, "\Uparrow 4."] & & CA'B \arrow[u, phantom, shift right=1, "\Uparrow 5."] \\
& A'CB \arrow[ur] &
\end{tikzcd}
$$

1\. $= \otimes_{(f,R_{B,C})}$ 2. $= R_{f,C} \otimes B$ 3. $= \tilde{R}_{(A,B|C)}$ 4. $= \tilde{R}_{(A',B|C)}$ 5. $= R_{f\otimes B,C}$

$((\bullet\otimes \to) \otimes \bullet)$, $(\bullet \otimes (\to \otimes\bullet))$ *and* $(\bullet \otimes (\bullet\otimes \to))$ *Similar prisms, left to the reader.*

$((\bullet \otimes \bullet \otimes \bullet) \otimes \bullet)$, $(\bullet \otimes (\bullet \otimes \bullet \otimes \bullet))$, $((\bullet \otimes \bullet) \otimes (\bullet \otimes \bullet))$ *As in Definition 6.*

$S^+ = S^-$ *As in Definition 6.*

Proof - The 1-equivalences $R_{A,B}$ and 2-isomorphisms $R_{f,B}$ and $R_{A,g}$ comprise the pseudonatural equivalence $R\colon \otimes \to \otimes^{\mathrm{op}}$, and conditions $(\to \otimes \to)$, $(\bullet\otimes \Downarrow)$, $(\Downarrow \otimes\bullet)$, $(\to\to \otimes\bullet)$ and $(\bullet\otimes \to\to)$ state that it is indeed a pseudonatural transformation. The 2-morphisms $\tilde{R}_{(A|B,C)}$ and $\tilde{R}_{(A,B|C)}$ comprise the invertible modifications $\tilde{R}_{(\text{-}|\text{-},\text{-})}$ and $\tilde{R}_{(\text{-},\text{-}|\text{-})}$, and the commuting triangular prisms state that these are indeed modifications, expressing naturality in each argument. The remaining 4 conditions come from Definition 6. $\Box$

Note that by $(\to\to \otimes\bullet)$ resp. $(\bullet\otimes \to\to)$ and by the invertibility of the respective 2-morphisms, for any objects $A, B \in \mathcal{C}$ we have $R_{A,1_B} = 1_{R_{A,B}}$ and $R_{1_A,B} = 1_{R_{A,B}}$.

The above lemma makes it clear that our definition of braided monoidal 2-category differs from that of Kapranov and Voevodsky in precisely the following points:

1. Invertibility of the braiding. Our definition implies that the 1-morphisms $R_{A,B}$ are equivalences. Kapranov and Voevodsky make no invertibility assumptions on these 1-morphisms. Our definition would agree with theirs on this point, and otherwise stay the same, if we required $R\colon \otimes \to \otimes^{\mathrm{op}}$ to be merely a pseudonatural transformation, rather than a pseudonatural equivalence.

2. $S^+ = S^-$. As already noted, Kapranov and Voevodsky omit this condition.

3. Naturality of $\tilde{R}_{(\text{-}|\text{-},\text{-})}$ and $\tilde{R}_{(\text{-},\text{-}|\text{-})}$. Our definition implies the commutativity of 6 triangular prisms expressing the naturality in each argument of these modifications. Kapranov and Voevodsky substitute cubes for 4 of these prisms, namely $(\bullet \otimes (\rightarrow \otimes \bullet))$, $(\bullet \otimes (\bullet \otimes \rightarrow))$, $((\bullet \otimes \rightarrow) \otimes \bullet)$ and $((\rightarrow \otimes \bullet) \otimes \bullet)$. By the following lemma one can deduce these cubes from the remaining data — but not, it appears, vice versa. In personal communication, Kapranov agreed that all these prisms should hold.

Lemma 8. *For any three objects $A, B, C \in \mathcal{C}$ and any morphism $f: B \to B'$, the following cube commutes.*

$$
\begin{tikzcd}
 & ABC \arrow[rr] \arrow[dl] \arrow[dd, dashed] & & AB'C \arrow[dl] \arrow[dd] \arrow[dl, phantom, "1."] \\
ACB \arrow[rr] \arrow[dd] & & ACB' \arrow[dd] \arrow[r, phantom, "6."] & {} \\
 & BCA \arrow[rr, dashed] \arrow[dl, dashed] \arrow[dl, phantom, "4."] \arrow[d, phantom, "2."] & & B'CA \arrow[dl] \arrow[dl, phantom, "3."] \\
CBA \arrow[rr] & {} & CB'A \arrow[u, phantom, "5."] &
\end{tikzcd}
$$

$$
\begin{array}{llllll}
1. = & A \otimes R_{f,C} & 2. = & R_{f,C} \otimes A & 3. = & R_{A,R_{B',C}} \\
4. = & R_{A,R_{B,C}} & 5. = & R_{A,C\otimes f} & 6. = & R_{A,f\otimes C}
\end{array}
$$

Proof - This is an special case of the axiom $(\bullet\otimes \Downarrow)$ together with $(\bullet\otimes \rightarrow\rightarrow)$. □

We refer to this cube with the hieroglyph $(\bullet\otimes(\rightarrow \otimes\bullet))'$. One can similarly prove the analogous cube corresponding to the hieroglyph $(\bullet\otimes(\bullet\otimes \rightarrow))'$ commutes. Moreover, we can prove the commutativity of cubes corresponding to the hieroglyphs $((\bullet\otimes \rightarrow) \otimes \bullet)'$ and $((\rightarrow \otimes\bullet) \otimes \bullet)'$ using $(\Downarrow \otimes\bullet)$ and $(\rightarrow\rightarrow \otimes\bullet)$.

3 The Center Construction

Let $(\mathcal{C}, \otimes, 1)$ be a semistrict monoidal 2-category. The center $\mathcal{Z}(\mathcal{C})$ would be easy to construct if we had a properly functioning theory of semistrict weak 4-categories. As it stands, all we can do is use our limited insight into 4-categories to guess the right answer, and then try to justify it by proving that we obtain a braided monoidal 2-category with good properties. We proceed in several stages. First we describe $\mathcal{Z}(\mathcal{C})$ as a 2-category. Then we describe the monoidal structure, and then the braiding.

3.1 $\mathcal{Z}(\mathcal{C})$ as a 2-Category

As noted in Section 1.1, the center construction applied to a monoid yields its usual center, because a certain square must commute. However, as one would expect from the weakening principle, when $\mathcal{C}$ is a monoidal category the corresponding square need only commute *up to a specified natural isomorphism.* An object of $\mathcal{Z}(\mathcal{C})$ thus turns out to be an object $A \in \mathcal{C}$ equipped with a natural isomorphism $R_{A,\text{-}}\colon A \otimes \text{-} \Rightarrow \text{-} \otimes A$ satisfying various coherence laws, such as the commutativity of following diagram:

$$\begin{tikzcd}
A \otimes X \otimes Y \arrow[rr, "R_{A,X\otimes Y}"] \arrow[dr, "R_{A,X} \otimes Y"'] & & X \otimes Y \otimes A \\
& X \otimes A \otimes Y \arrow[ur, "X \otimes R_{A,Y}"'] &
\end{tikzcd}$$

for any objects $X, Y \in \mathcal{C}$. Of course, this diagram is part of the definition of a braided monoidal category. Similarly, the morphisms in $\mathcal{Z}(\mathcal{C})$ also work out to have properties that form part of the definition of a braided monoidal category.

Heuristic 4-categorical computations suggest how these patterns should continue when $\mathcal{C}$ is a monoidal 2-category. We thus define $\mathcal{Z}(\mathcal{C})$ as follows.

Objects in $\mathcal{Z}(\mathcal{C})$:

An object of $\mathcal{Z}(\mathcal{C})$ is a triple $(A, R_{A,\text{-}}, \tilde{R}_{(A|\text{-},\text{-})})$ consisting of:

1. an object $A \in \mathcal{C}$
2. a pseudonatural equivalence $R_{(A,\text{-})}\colon A \otimes \text{-} \Rightarrow \text{-} \otimes A$
3. an invertible modification $\tilde{R}_{(A|\text{-},\text{-})}$, giving for any objects $X, Y \in \mathcal{C}$ a 2-isomorphism

$$\begin{tikzcd}
A \otimes X \otimes Y \arrow[rr, "R_{A,X\otimes Y}"] \arrow[dr, "R_{A,X} \otimes Y"'] & \Uparrow \tilde{R}_{(A|X,Y)} & X \otimes Y \otimes A \\
& X \otimes A \otimes Y \arrow[ur, "X \otimes R_{A,Y}"'] &
\end{tikzcd}$$

such that for any objects $X, Y, Z \in \mathcal{C}$, the tetrahedron $(\bullet \otimes (\bullet \otimes \bullet \otimes \bullet))$ commutes.

Here we mean that the diagram $(\bullet \otimes (\bullet \otimes \bullet \otimes \bullet))$ commutes with objects A, X, Y, Z, and with the modification $\tilde{R}_{(\text{-}|\text{-},\text{-})}$ in the definition of a braided monoidal 2-category replaced by the above $\tilde{R}_{(A|\text{-},\text{-})}$. Throughout the following we use the hieroglyphical notation in this way. Also, we use letters near the beginning of the alphabet to denote objects of $\mathcal{C}$ underlying objects in $\mathcal{Z}(\mathcal{C})$, and letters near the end to denote objects of $\mathcal{C}$ being used as such.

Remark 9. *The fact that $R_{A,\text{-}}$ is a pseudonatural equivalence can be expressed equivalently as follows: for any object $X \in \mathcal{C}$, there exists an equivalence $R_{A,X}\colon A \otimes X \to X \otimes A$, and for any morphism $f\colon X \to Y$ in $\mathcal{C}$, there exists a 2-isomorphism $R_{A,f}\colon (A \otimes f) \circ R_{A,Y} \Rightarrow R_{A,X} \circ (f \otimes A)$:*

$$\begin{tikzcd}
A \otimes X \arrow[r, "R_{A,X}"] \arrow[d, "A\otimes f"'] \arrow[dr, phantom, "\Uparrow R_{A,f}"] & X \otimes A \arrow[d, "f\otimes A"] \\
A \otimes Y \arrow[r, "R_{A,Y}"'] & Y \otimes A
\end{tikzcd}$$

such that $(\bullet\otimes \to\to)$ and $(\bullet\otimes \Downarrow)$ commute.

Similarly, the fact that $\tilde{R}_{(A|\text{-},\text{-})}$ is a modification means that the diagrams $(\bullet \otimes (\to \otimes\bullet))$ and $(\bullet \otimes (\bullet\otimes \to))$ commute.

Morphisms in $\mathcal{Z}(\mathcal{C})$:

A morphism in $\mathcal{Z}(\mathcal{C})$ from $(A, R_{A,\text{-}}, \tilde{R}_{(A|\text{-},\text{-})})$ to $(B, R_{B,\text{-}}, \tilde{R}_{(B|\text{-},\text{-})})$ is a pair $(f, R_{f,\text{-}})$ consisting of:

1. a morphism $f\colon A \to B$

2. an invertible modification $R_{f,\text{-}}$, giving for any object $X \in \mathcal{C}$ a 2-isomorphism

$$\begin{tikzcd}
A \otimes \text{-} \arrow[r, "f \otimes \text{-}"] \arrow[d, "R_{A,\text{-}}"'] \arrow[dr, phantom, "\Downarrow R_{f,\text{-}}"] & B \otimes \text{-} \arrow[d, "R_{B,\text{-}}"] \\
\text{-} \otimes A \arrow[r, "\text{-} \otimes f"'] & \text{-} \otimes B
\end{tikzcd}$$

such that the prism $(\to \otimes(\bullet \otimes \bullet))$ commutes.

Remark 10. *The fact that $R_{f,\text{-}}$ is a modification can be expressed equivalently by saying that $(\to \otimes \to)$ commutes. (Note that $(f \otimes \text{-})R_{B,\text{-}}$ and $R_{A,\text{-}}(\text{-} \otimes f)$ are pseudonatural transformations in an obvious way.)*

2-Morphisms in $\mathcal{Z}(\mathcal{C})$:

A 2-morphism α in $\mathcal{Z}(\mathcal{C})$ from $(f, R_{f,\text{-}})$ to $(g, R_{g,\text{-}})$ is

1. a 2-morphism $\alpha: f \Rightarrow g$ in $\mathcal{C}$

such that $(\Downarrow \otimes \bullet)$ commutes.

We define the composition operations in $\mathcal{Z}(\mathcal{C})$ as follows. Composition of morphisms is defined by:

$$(f, R_{f,\text{-}}) \circ (g, R_{g,\text{-}}) := (f \circ g, ((f \otimes \text{-}) \circ R_{g,\text{-}}) \cdot (R_{f,\text{-}} \circ (\text{-} \otimes g)))$$

where $f: A \to B$ and $g: B \to C$ are the underlying 1-morphisms in $\mathcal{C}$. Note that for any object $X \in \mathcal{C}$, the 2-morphism $((f \otimes X) \circ R_{g,X}) \cdot (R_{f,X} \circ (X \otimes g))$ equals the back of the following diagram:

$$\begin{tikzcd}
AX \arrow[rr, "(f\circ g)\otimes X"] \arrow[dr, "f\otimes X"'] \arrow[dd, "R_{A,X}"'] \arrow[r, phantom, "\Uparrow\mathrm{id}"] & & CX \arrow[dd, "R_{C,X}"] \\
\arrow[r, phantom, "\Downarrow R_{f,X}"] & BX \arrow[ur, "g\otimes X"'] \arrow[dd] \arrow[r, phantom, "\Downarrow R_{g,X}"] & \\
XA \arrow[rr, dashed] \arrow[dr, "X\otimes f"'] \arrow[r, phantom, "\Uparrow\mathrm{id}"] & & XC \\
& XB \arrow[ur, "X\otimes g"'] &
\end{tikzcd}$$

Remark 11. *Eventually this will imply that the braiding in $\mathcal{Z}(\mathcal{C})$ satisfies $(\to\to \otimes\bullet)$.*

To show that the composite of morphisms in $\mathcal{Z}(\mathcal{C})$ is again a morphism, we have to check that $(\to \otimes \to)$ and $(\to \otimes(\bullet \otimes \bullet))$ commute. These can be seen by pasting together two diagrams of the form $(\to \otimes \to)$ and $(\to \otimes(\bullet \otimes \bullet))$, respectively.

Vertical and horizontal composition of 2-morphisms is defined the same as in $\mathcal{C}$; one can check that these composites again satisfy $(\Downarrow \otimes\bullet)$ by pasting together two diagrams of this form.

3.2 The Monoidal Structure

We have to show that $\mathcal{Z}(\mathcal{C})$ bears a monoidal structure $(\mathcal{Z}(\mathcal{C}), \otimes_{\mathcal{Z}(\mathcal{C})}, I)$, such that all the requirements for a monoidal category given in Definition 4 are satisfied.

(Ad 4.1): The object $I \in \mathcal{Z}(\mathcal{C})$ is $(I, 1_{\text{-}}, 1_{1_{(\text{-}\otimes\text{-})}})$.

The tensor product of objects: (Ad 4.2): The tensor product of two objects $(A, R_{A,\text{-}}, \tilde{R}_{(A|\text{-},\text{-})}) \otimes_{\mathcal{Z}(\mathcal{C})} (B, R_{B,\text{-}}, \tilde{R}_{(B|\text{-},\text{-})}$ is defined to be the triple $(A \otimes B, (R_A \otimes R_B)_{\text{-}}, (\tilde{R}_A \otimes \tilde{R}_B)_{(\text{-},\text{-})})$, where:

1. The underlying $\mathcal{C}$-object is the tensor product $A \otimes B$ in $\mathcal{C}$.

2. By Remark (9), the underlying pseudonatural equivalence $(R_A \otimes R_B)_{\text{-}} : (A \otimes B) \otimes \text{-} \Rightarrow \text{-} \otimes (A \otimes B)$ assigns a 1-morphism $(R_A \otimes R_B)_X$ to any object $X \in \mathcal{C}$ and a 2-morphism $(R_A \otimes R_B)_f$ to any 1-morphism $f \colon X \to Y$. These are given as follows:
$$(R_A \otimes R_B)_X = (A \otimes R_{B,X})(R_{A,X} \otimes B),$$
$$(R_A \otimes R_B)_f = ((A \otimes R_{B,f}) \circ (R_{A,Y} \otimes B)) \cdot (A \otimes R_{B,X}) \circ (R_{A,f} \otimes B)),$$
or in terms of a diagram:
$$\begin{tikzcd}
ABX \arrow[d, "AB\otimes f"'] \arrow[drr, "A\otimes R_{B,X}"] & & & & XAB \arrow[d, "XA\otimes f"] \\
 & & AXB \arrow[urr, "R_{A,X}\otimes B"] \arrow[dd] & & \\
ABY \arrow[urr, phantom, "\Uparrow A\otimes R_{B,f}"] \arrow[drr, "A\otimes R_{B,Y}"'] & & & & YAB \arrow[ull, phantom, "\Uparrow R_{A,f}\otimes B"'] \\
 & & AYB \arrow[urr, "R_{A,Y}\otimes B"'] & &
\end{tikzcd}$$

   Remark 12. *This will imply that the braiding in $\mathcal{Z}(\mathcal{C})$ satisfies $((\bullet \otimes \bullet)\otimes \rightarrow)$.*

   To show that these data constitute a pseudonatural equivalence, we have to show that $(\bullet\otimes \rightarrow\rightarrow)$ and $(\bullet\otimes \Downarrow)$ hold. This can be done easily by pasting together the corresponding diagrams for $R_{A,\text{-}}$ and $R_{B,\text{-}}$.

3. The underlying modification $(\tilde{R}_A \otimes \tilde{R}_B)_{(X,Y)} : (A \otimes R_{B,X} \otimes Y)(R_{A,X} \otimes B \otimes Y)(X \otimes A \otimes R_{B,Y})(X \otimes R_{A,Y} \otimes B) \Rightarrow (A \otimes R_{B,X\otimes Y})(R_{A,X\otimes Y} \otimes B)$ is defined to be the pasting:

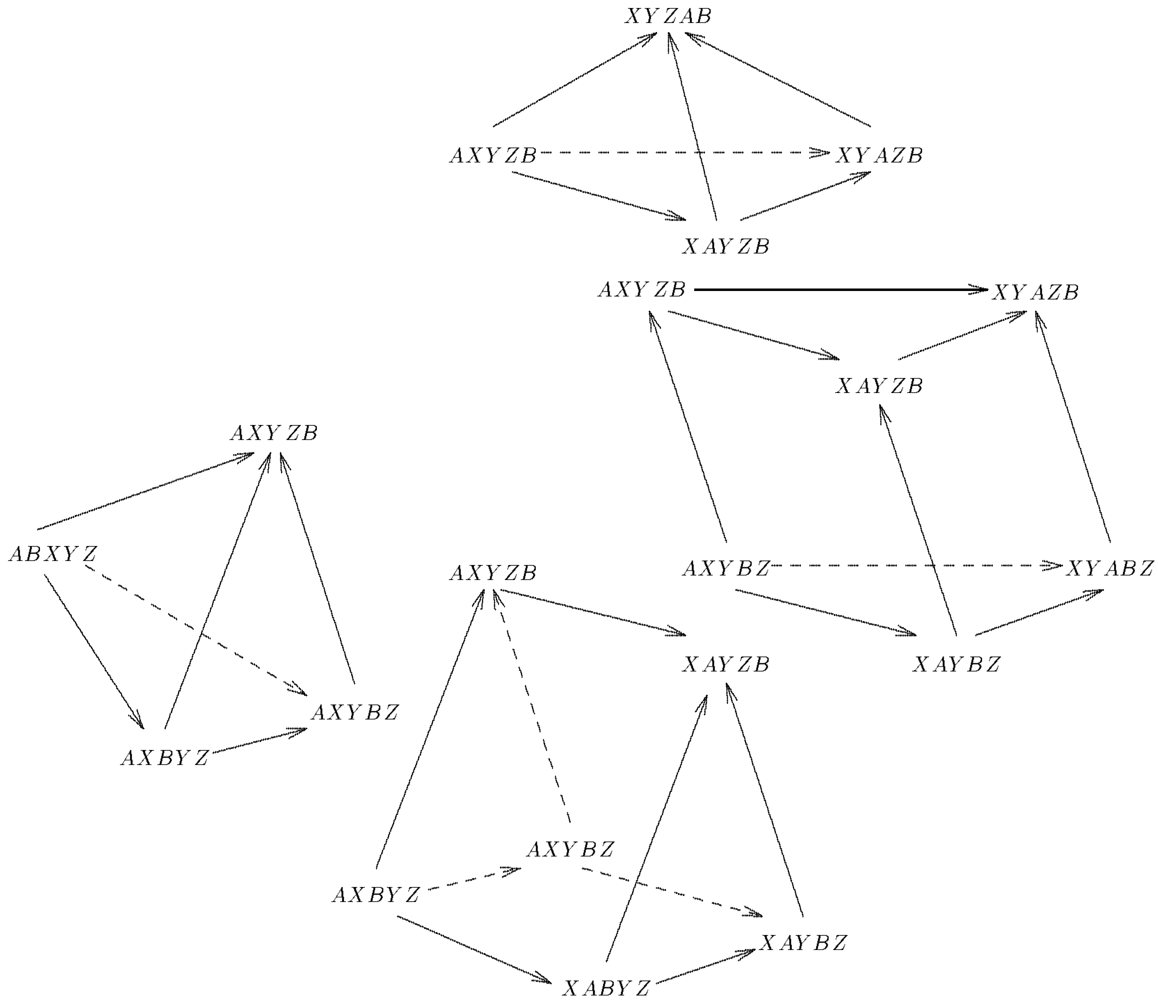
$$
\begin{tikzcd}
ABXY \arrow[drrr, "A\otimes R_{B,XY}"] \arrow[ddr, "A\otimes R_{B,X}\otimes Y"'] & & & & & & XYAB \\
& \arrow[r, phantom, "\Uparrow A\otimes \tilde{R}_{(B|X,Y)}"] & & AXYB \arrow[urrr, "R_{A,XY}\otimes B"] \arrow[drr] & \arrow[r, phantom, "\Uparrow \tilde{R}_{(A|X,Y)}\otimes B"] & & \\
& AXBY \arrow[urr] \arrow[drr, "R_{A,X}\otimes BY"'] \arrow[rrrr, phantom, "\Uparrow \otimes^{-1}_{R_{A,X},R_{B,Y}}"] & & & & XAYB \arrow[uur, "X\otimes R_{A,Y}\otimes B"'] & \\
& & & XABY \arrow[urr, "XA\otimes R_{B,Y}"'] & & &
\end{tikzcd}
$$

Again it is easy to verify that this satisfies $(\bullet\otimes(\rightarrow\otimes\bullet))$ and $(\bullet\otimes(\bullet\otimes\rightarrow))$ and hence is a modification.

To show that this definition gives in fact an object in $\mathcal{Z}(\mathcal{C})$, we have to verify that $(\bullet\otimes(\bullet\otimes\bullet\otimes\bullet))$ is satisfied. The following picture shows the tetrahedron. Those vertices in the picture that are vertices of the tetrahedron are written in big capitals. The remaining vertices occur since they are needed for the decomposition.

$$
\begin{tikzcd}
& & & XYZAB & & & \\
& AXYZB \arrow[urr] & & & & XYAZB \arrow[ull] & \\
& & & XAYZB \arrow[uu] & & & \\
ABXYZ \arrow[uur] \arrow[r, dashed] \arrow[dr] & AXYBZ \arrow[rrrrr, dashed] & & & & & XYABZ \arrow[uul] \\
& AXBYZ \arrow[drr] & & & & XAYBZ \arrow[ur] & \\
& & & XABYZ \arrow[uuu] \arrow[urr] & & &
\end{tikzcd}
$$

The following picture gives a decomposition of the tetrahedron into four smaller commutative diagrams.



Two of them are tetrahedra of the form $(\bullet \otimes (\bullet \otimes \bullet \otimes \bullet))$, tensored by an object from the left and the right, respectively. The upper of the two triangular prisms commutes by the axioms 4.(vii) with $\alpha = \tilde{R}_{(A|X,Y)}$ and $g = R_{Z,B}$, together with 4.$(viii)$. The lower commutes by 4.(vi), applied to $\beta = \tilde{R}_{(B|Y,Z)}$ and $f = R_{A,X}$ together with 4.$(viii)$. One can verify that this tensor product is in fact associative.

We shall often write $(A \otimes B, R_A \otimes R_B, \tilde{R}_A \otimes \tilde{R}_B)$ as a shorthand symbol for the tensor product of objects in $\mathcal{Z}(\mathcal{C})$.

The tensor product of an object and a morphism:

(Ad 4.3): Let $(f, R_{f,\text{-}})\colon (A, R_{A,\text{-}}, \tilde{R}_{(A|\text{-},\text{-})}) \to (A', R_{A',\text{-}}, \tilde{R}_{(A'|\text{-},\text{-})})$ be a morphism in $\mathcal{Z}(\mathcal{C})$ and let $(B, R_{B,\text{-}}, \tilde{R}_{(B|\text{-},\text{-})})$ be an object. Their tensor product is the morphism given by the pair

$$(f \otimes B, (\otimes_{(f,R_{B,\text{-}})} \circ (R_{A',\text{-}} \otimes B)) \cdot ((A \otimes R_{B,\text{-}}) \circ (R_{f,\text{-}} \otimes B))),$$

or in terms of a diagram:

$$
\begin{tikzcd}
ABX \arrow[dr, "A\otimes R_{B,X}"] \arrow[dd, "f\otimes BX"'] & & XAB \arrow[dd, "X\otimes f\otimes B"] \\
& AXB \arrow[ur, "R_{A,X}\otimes B"] \arrow[dd] & \\
A'BX \arrow[dr, "A'\otimes R_{B,X}"'] \arrow[ur, phantom, "\Uparrow\otimes_{(f,R_{B,X})}"] & & XA'B \arrow[ul, phantom, "\Uparrow R_{f,X}\otimes B"'] \\
& A'XB \arrow[ur, "R_{A',X}\otimes B"'] &
\end{tikzcd}
$$

Remark 13. *This will imply that the braiding in $\mathcal{Z}(\mathcal{C})$ satisfies* $((\to\otimes\bullet)\otimes\bullet)$.

(Ad 4.4) Let $(f, R_{f,\text{-}}) : (B, R_{B,\text{-}}, \tilde{R}_{(B|\text{-},\text{-})}) \to (B', R_{B',\text{-}}, \tilde{R}_{(B'|\text{-},\text{-})})$ be a morphism in $\mathcal{Z}(\mathcal{C})$ and let $(A, R_{A,\text{-}}, \tilde{R}_{(A|\text{-},\text{-})})$ be an object. Their tensor product is the pair

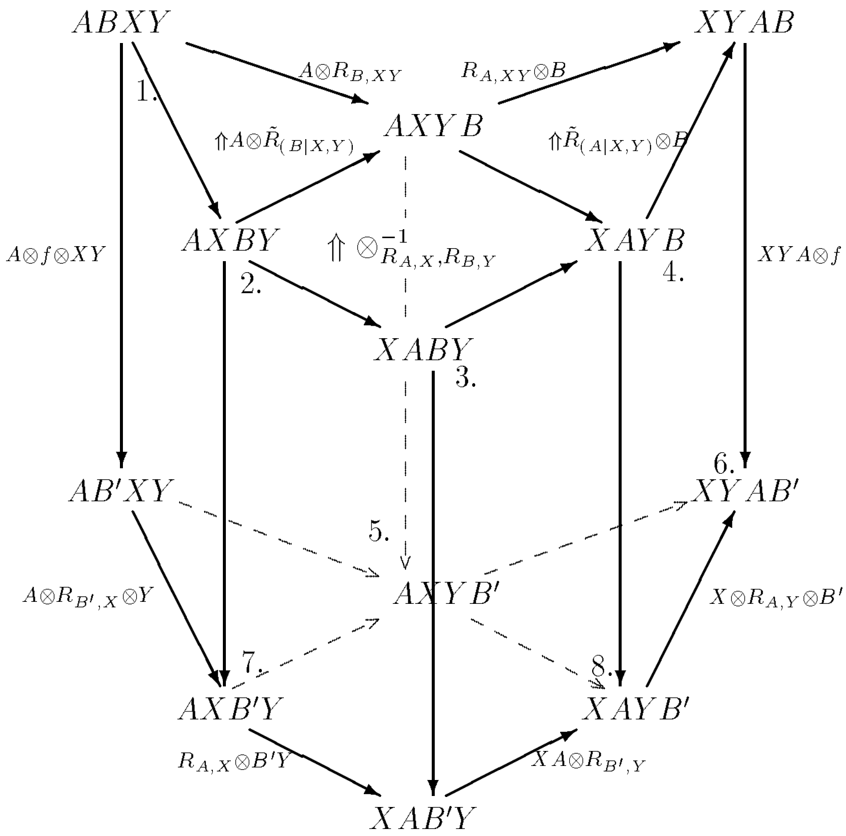
$$(A\otimes f, ((A\otimes R_{f,\text{-}})\circ(R_{A,\text{-}}\otimes B'))\cdot((A\otimes R_{B,\text{-}})\circ\otimes_{R_{A,\text{-}},f})),$$

or in terms of a diagram:

$$
\begin{tikzcd}
ABX \arrow[dr, "AR_{(B,X)}"] \arrow[dd, "A\otimes f\otimes X"'] & & XAB \arrow[dd, "XA\otimes f"] \\
& AXB \arrow[ur, "R_{(A,X)}B"] \arrow[dd] & \\
AB'X \arrow[dr, "AR_{(B',X)}"'] \arrow[ur, phantom, "\Uparrow A\otimes R_{f,X}"] & & XAB' \arrow[ul, phantom, "\Uparrow\otimes_{R_{A,X},f}"'] \\
& AXB' \arrow[ur, "R_{(A,X)}B'"'] &
\end{tikzcd}
$$

Remark 14. *This will imply that the braiding in $\mathcal{Z}(\mathcal{C})$ satisfies* $((\bullet\otimes\to)\otimes\bullet)$.

To verify that these formulas really define morphisms in $\mathcal{Z}(\mathcal{C})$, one must check that $(\to\otimes\to)$ and $(\to\otimes(\bullet\otimes\bullet))$ hold. We only do this for 4.4; the other case being similar. To show $(\to\otimes\to)$ one pastes together two cubes, one being the $(\to\otimes\to)$ cube for $f\colon B\to B'$ and $g\colon X\to Y$ tensored on the left by A, the other being a special case of 5.(vii). For $(\to\otimes(\bullet\otimes\bullet))$ we must show the following diagram commutes:



1. $= A\otimes R_{f,X}\otimes Y$	2. $= \otimes_{R_{A,X},f\otimes Y}$	3. $= X\otimes A\otimes R_{f,Y}$	4. $= X\otimes\otimes_{R_{A,Y},f}$
5. $= A\otimes R_{f,X\otimes Y}$	6. $= \otimes_{R_{A,X\otimes Y},f}$	7. $= A\otimes X\otimes R_{f,Y}$	8. $= \otimes_{R_{A,X},Y\otimes f}$

We cut it into one rectangular and two triangular prisms. To see that the left triangular prism commutes, we apply $(\to\otimes(\bullet\otimes\bullet))$ to $(f, R_{f,\text{-}})$, tensored on the left by A. The rectangular prism commmutes by 5.(vi) and 5.$(viii)$, applied to the 2-morphism $R_{f,Y}$. The right triangular prism commutes by 5.(iv) and 5.(vii), applied to the 2-morphism $\tilde{R}_{(A|X,Y)}$.

The tensor product of an object and a 2-morphism:

(Ad 4.5): For any object $(A, R_{A,\text{-}}, \tilde{R}_{(A|\text{-},\text{-})})$ and any 2-morphism $\alpha\colon(f, R_{f,\text{-}})\Rightarrow(f', R_{f',\text{-}})$ we have a 2-morphism

$$A\otimes\alpha : (A\otimes f,\ldots)\Rightarrow(A\otimes f',\ldots)$$

(Ad 4.6): For any object $(B, R_{B,\text{-}}, \tilde{R}_{(B|\text{-},\text{-})})$ and any 2-morphism $\alpha : (g, R_{g,\text{-}})\Rightarrow(g', R_{g',\text{-}})$ we have a 2-morphism

$$\alpha\otimes B : (g\otimes B,\ldots)\Rightarrow(g'\otimes B,\ldots)$$

We must verify that these are 2-morphisms in $\mathcal{Z}(\mathcal{C})$, so we must check $(\Downarrow \otimes \bullet)$. We do this only for 4.5.

$$
\begin{tikzcd}
ABX \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow_{A\otimes\alpha\otimes X}"] \arrow[d, "A\otimes R_{B,X}"'] \arrow[dr, phantom, "\text{2.}", pos=0.2] \arrow[dr, phantom, "\text{1.}", pos=0.8] & AB'X \arrow[d, "A\otimes R_{B',X}"] \\
AXB \arrow[r, dashed, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow_{AX\otimes\alpha}"] \arrow[d, "R_{A,X}\otimes B"'] \arrow[dr, phantom, "\text{4.}", pos=0.2] \arrow[dr, phantom, "\text{3.}", pos=0.8] & AXB' \arrow[d, "R_{A,X}\otimes B'"] \\
XAB \arrow[r, dashed, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow_{XA\otimes\alpha}"] & XAB'
\end{tikzcd}
$$

$$
\begin{array}{ll}
1. = A\otimes R_{f,X} & 2. = A\otimes R_{g,X} \\
3. = \otimes_{R_{A,X},f} & 4. = \otimes_{R_{A,X},g}
\end{array}
$$

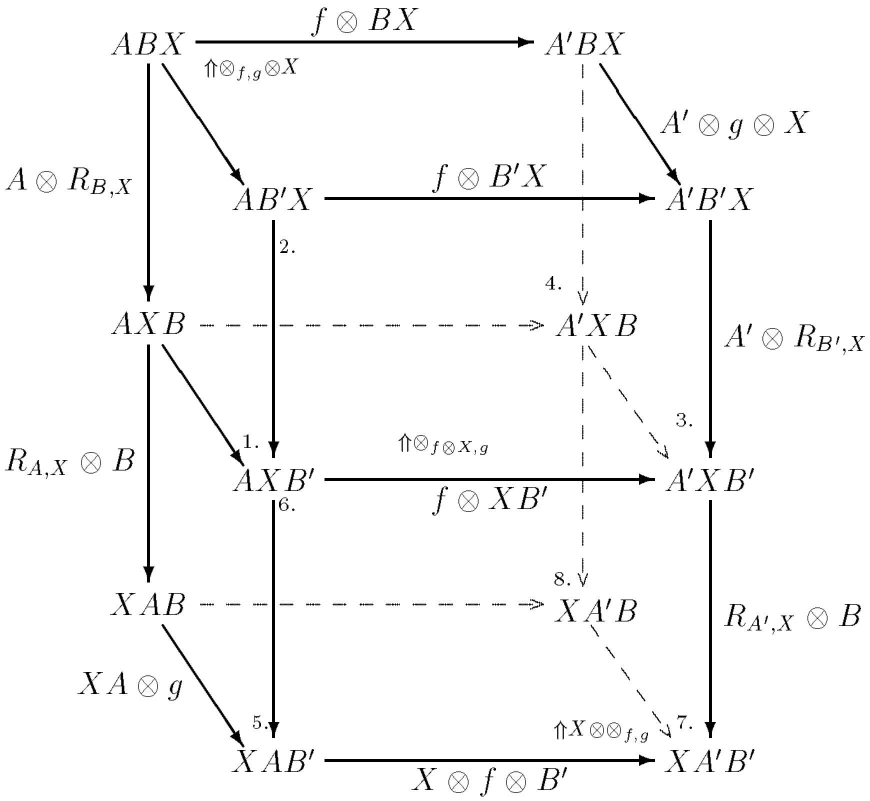
The upper prism commutes by $(\Downarrow \otimes\bullet)$ tensored from the left by A. The lower prism commutes by an application of the axiom $4.(vi)$ for monoidal 2-categories to the 2-morphism α.

The tensor product of morphisms:

(Ad 4.7): For any morphisms $(f, R_{f,\text{-}}) : (A, R_A, \tilde{R}_A) \to (A', R_{A'}, \tilde{R}_{A'})$ and $(g, R_{g,\text{-}}) : (B, R_B, \tilde{R}_B) \to (B', R_{B'}, \tilde{R}_{B'})$ we have a 2-isomorphism:

$$\otimes_{(f,R_{f,\text{-}}),(g,R_{g,\text{-}})} := \otimes_{f,g}$$

To verify that this is a 2-morphism in $\mathcal{Z}(\mathcal{C})$, we have to check $(\Downarrow \otimes\bullet)$. The following diagram gives the proof.



1. $= A \otimes R_{g,X}$	2. $= \otimes_{f,R_{B',X}}$	3. $= A' \otimes R_{g,X}$	4. $= \otimes_{f,R_{B,X}}$
5. $= \otimes_{R_{A,X},g}$	6. $= R_{f,X} \otimes B'$	7. $= \otimes_{R_{A',X},g}$	8. $= R_{f,X} \otimes B$

The top cube commutes by $4.(iv), (vi), (viii)$, applied to the 2-morphism $A \otimes R_{g,X}$. The bottom cube commutes by $4.(iv), (vii), (viii)$, applied to the 2-morphism $R_{f,X} \otimes B$.

We have to verify that these data satisfy the conditions $4.(i) - (viii)$. These follow from the corresponding conditions holding in $\mathcal{C}$.

3.3 The Braiding

$(\bullet \otimes \bullet)$: For any two objects we have the morphism

$$(R_{A,B}, R_{R_{A,B},\text{-}}) : (A \otimes B, R_{A,\text{-}} \otimes R_{B,\text{-}}, \tilde{R}_A \otimes \tilde{R}_B) \to (B \otimes A, R_{B,\text{-}} \otimes R_{A,\text{-}}, \tilde{R}_B \otimes \tilde{R}_A)$$

in $\mathcal{Z}(\mathcal{C})$, where the 2-morphism $R_{R_{A,B},X}$ is defined to be the pasting:

$$
\begin{tikzcd}
ABX \arrow[r, "R_{A,B}\otimes X"] \arrow[d, "A\otimes R_{B,X}"'] \arrow[dr] & BAX \arrow[d, "B\otimes R_{A,X}"] \arrow[dl, phantom, "\Downarrow\tilde{R}_{(A|B,X)}"] \\
AXB \arrow[r, phantom, "\Downarrow R^{-1}_{A,R_{B,X}}"] \arrow[d, "R_{A,X}\otimes B"'] \arrow[dr] & BXA \arrow[d, "R_{B,X}\otimes A"] \\
XAB \arrow[r, "X\otimes R_{A,B}"'] \arrow[ur, phantom, "\Downarrow\tilde{R}^{-1}_{(A|X,B)}"] & XBA
\end{tikzcd}
$$

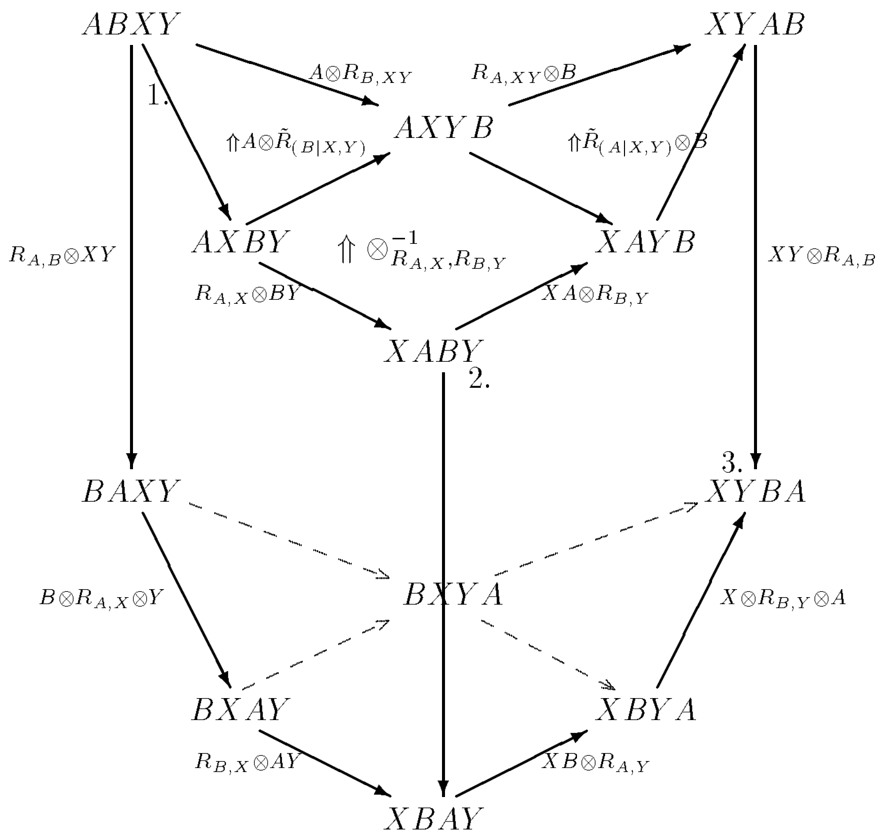
First we have to show that $R_{(R_{A,B},\text{-})}$ satisfies $(\rightarrow\otimes\rightarrow)$ and hence is a modification. This is shown in the following diagram (or follows from the fact that it is a pasting of modifications).

$$
\begin{tikzcd}
ABX \arrow[rr, "R_{A,B}\otimes X"] \arrow[dd, "A\otimes R_{B,X}"'] \arrow[dr] \arrow[ddrr, dashed] & & BAX \arrow[dr, "BA\otimes f"] \arrow[dd, dashed] & \\
& ABX' \arrow[rr, "R_{A,B}\otimes X'"] \arrow[dd] \arrow[ddrr] & & BAX' \arrow[dd, "B\otimes R_{A,X'}"] \\
AXB \arrow[dd, "R_{A,X}\otimes B"'] \arrow[dr] \arrow[ddrr, dashed] & & BXA \arrow[dr, dashed] \arrow[dd, dashed] & \\
& AX'B \arrow[dd] \arrow[ddrr] & & BX'A \arrow[dd, "R_{B,X'}\otimes A"] \\
XAB \arrow[rr, dashed] \arrow[dr, "f\otimes AB"'] & & XBA \arrow[dr, dashed] & \\
& X'AB \arrow[rr, "X'\otimes R_{A,B}"'] & & X'BA
\end{tikzcd}
$$

The front and the back side of the cube are the 2-morphisms $R_{R_{A,B},X}$ and $R_{R_{A,B},X'}$, respectively. The top and the bottom are $\otimes_{R_{A,B},f}$ and $\otimes_{f,R_{A,B}}$, respectively. The left and the right side are the 2-morphisms corresponding to the pseudonatural transformations in the tensor product of the objects A and B, $(R_A\otimes R_B)_f$ and $(R_B\otimes R_A)_f$, respectively.

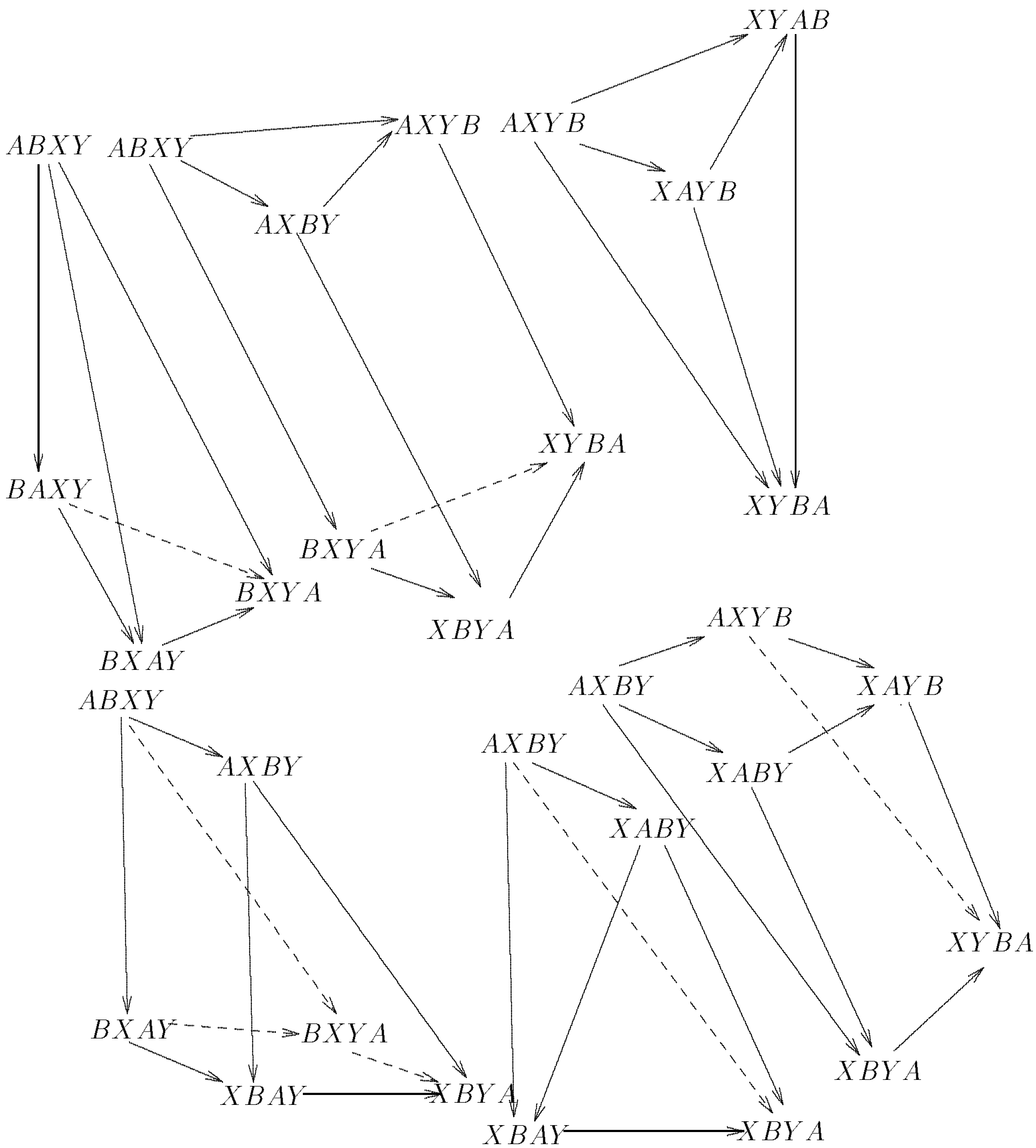
The top triangular prism commutes by $(\bullet \otimes (\bullet \otimes \rightarrow))$. The bottom triangular prism commutes by $(\bullet \otimes (\rightarrow \otimes \bullet))$. The cube in the middle commutes by $(\bullet \otimes (\bullet \otimes \rightarrow))'$, which is a consequence of $(\bullet \otimes \Downarrow)$ and $(\bullet \otimes \rightarrow\rightarrow)$ as indicated in Lemma 8.

Next, to show that we have really defined a morphism in $\mathcal{Z}(\mathcal{C})$, we have to verify $(\rightarrow \otimes (\bullet \otimes \bullet))$. This means we have to check the commutativity of the following diagram.



$$1. = R_{R_{A,B},X} \otimes Y \qquad 2. = X \otimes R_{R_{A,B},Y} \qquad 3. = R_{R_{A,B},X \otimes Y)}$$

As shown in the diagram below, we decompose this diagram in the following way: 1) Three tetrahedra of the form $(\bullet \otimes (\bullet \otimes \bullet \otimes \bullet))$. 2) One prism of the form $(\bullet \otimes (\bullet \otimes \rightarrow))$, namely $(A \otimes (X \otimes (BY \rightarrow YB)))$ (second row, right). 3) One prism of the form $(\bullet \otimes (\rightarrow \otimes \bullet))$, namely $(A \otimes ((BX \rightarrow XB) \otimes Y))$ (second row, left). 4) One prism of the form $(\bullet \otimes \Downarrow)$, namely $(A \otimes \tilde{R}_{(B|X,Y)}$ (in the middle of the first row). All of these diagrams commute by our assumptions.



($\to \otimes \bullet$): For any 1-morphism $(f, R_{f,\text{-}})\colon (A, R_A, \tilde{R}_A) \to (A', R_{A'}, \tilde{R}_{A'})$ and any object $(B, R_B, \tilde{R}_B) \in \mathcal{Z}(\mathcal{C})$ we have a 2-isomorphism

$$R_{f,B} : (f \otimes B)R_{A',B} \Rightarrow R_{A,B}(B \otimes f)$$

The following diagram shows that $R_{f,B}$ satisfies ($\Downarrow \otimes \bullet$) and is therefore a 2-morphism in $\mathcal{Z}(\mathcal{C})$.

$$
\begin{tikzcd}
ABX \arrow[rrr, "f \otimes BX"] \arrow[dr] \arrow[dd, "A \otimes R_{B,X}"'] \arrow[dddr] \arrow[rr, phantom, "\Downarrow R_{f,B} \otimes X"] & & & A'BX \arrow[dr, "R_{A',B} \otimes X"] \arrow[dd, dashed] \arrow[dddr, dashed] & & \\
& BAX \arrow[rrr, "B \otimes f \otimes X"] \arrow[dd] & & & BA'X \arrow[dd, "B \otimes R_{A',X}"] & (\to \otimes (\bullet \otimes \bullet)) \\
AXB \arrow[rrr, dashed] \arrow[dd, "R_{A,X} \otimes B"'] \arrow[dddr] & & & A'XB \arrow[dd, dashed] \arrow[dddr, dashed] & & \\
& BXA \arrow[rrr] \arrow[dd] & & & BXA' \arrow[dd, "R_{B,X} \otimes A'"] & (\to \otimes \to) \\
XAB \arrow[rrr, dashed] \arrow[dr, "X \otimes R_{A,B}"'] & & & XA'B \arrow[dr, dashed] & & \\
& XBA \arrow[rrr, "XB \otimes f"'] \arrow[rr, phantom, "\Downarrow X \otimes R_{f,B}"] & & & XBA' & (\to \otimes (\bullet \otimes \bullet))
\end{tikzcd}
$$

The left and right sides are the 2-morphisms $R_{R_{A,B},X}$ and $R_{R_{A',B},X}$, respectively. The front and the back sides are pastings as in our treatment in Section 3.2 of the tensor product of an object and a morphism in $\mathcal{Z}(\mathcal{C})$.

We decompose this cube into two commutative triangular prisms of the form $(\to \otimes(\bullet \otimes \bullet))$, correspoding to $(f \otimes (B \otimes X))$ and $(f \otimes (X \otimes B))$, and one cube of the form $(\to \otimes \to)$, namely $(A \to A' \otimes BX \to XB)$.

$(\bullet\otimes\ \to)$: For any 1-morphism $(g, R_{g,-}) : (B, R_B, \tilde{R}_B) \to (B', R_{B'}, \tilde{R}_{B'})$ and any object $(A, R_A, \tilde{R}_A) \in \mathcal{Z}(\mathcal{C})$, we have a 2-iso

$$R_{A,g} : (A \otimes g) R_{A,B'} \Rightarrow R_{A,B}(g \otimes A)$$

The following diagram shows that $R_{A,g}$ satisfies $(\Downarrow \otimes \bullet)$ and is thus a 2-morphism in $\mathcal{Z}(\mathcal{C})$.

$$
\begin{tikzcd}
ABX \arrow[rr, "A\otimes g\otimes X"] \arrow[dr] \arrow[dd, "A\otimes R_{B,X}"'] \arrow[dddr] \arrow[r, phantom, shift right=3ex, "\Downarrow R_{A,g}\otimes X"] & & AB'X \arrow[dr, "R_{A,B'}\otimes X"] \arrow[dd, dashed] \arrow[dddr, dashed] & & \\
& BAX \arrow[rr, "g\otimes AX"] \arrow[dd] & & B'AX \arrow[dd, "B'\otimes R_{A,X}"] & (\bullet\otimes(\rightarrow\otimes\bullet)) \\
AXB \arrow[rr, dashed] \arrow[dd, "R_{A,X}\otimes B"'] \arrow[dddr] & & AXB' \arrow[dd, dashed] \arrow[dddr, dashed] & & \\
& BXA \arrow[rr] \arrow[dd] & & B'XA \arrow[dd, "R_{B',X}\otimes A"] & \\
XAB \arrow[rr, dashed] \arrow[dr, "XR_{A,B}"'] & & XAB' \arrow[dr, dashed] & & (\bullet\otimes(\rightarrow\otimes\bullet))' \\
& XBA \arrow[rr, "X\otimes g\otimes A"'] \arrow[rr, phantom, shift left=3ex, "\Downarrow X\otimes R_{A,g}"] & & XB'A & (\bullet\otimes(\rightarrow\otimes\bullet))
\end{tikzcd}
$$

The decomposition is similar to the one before.

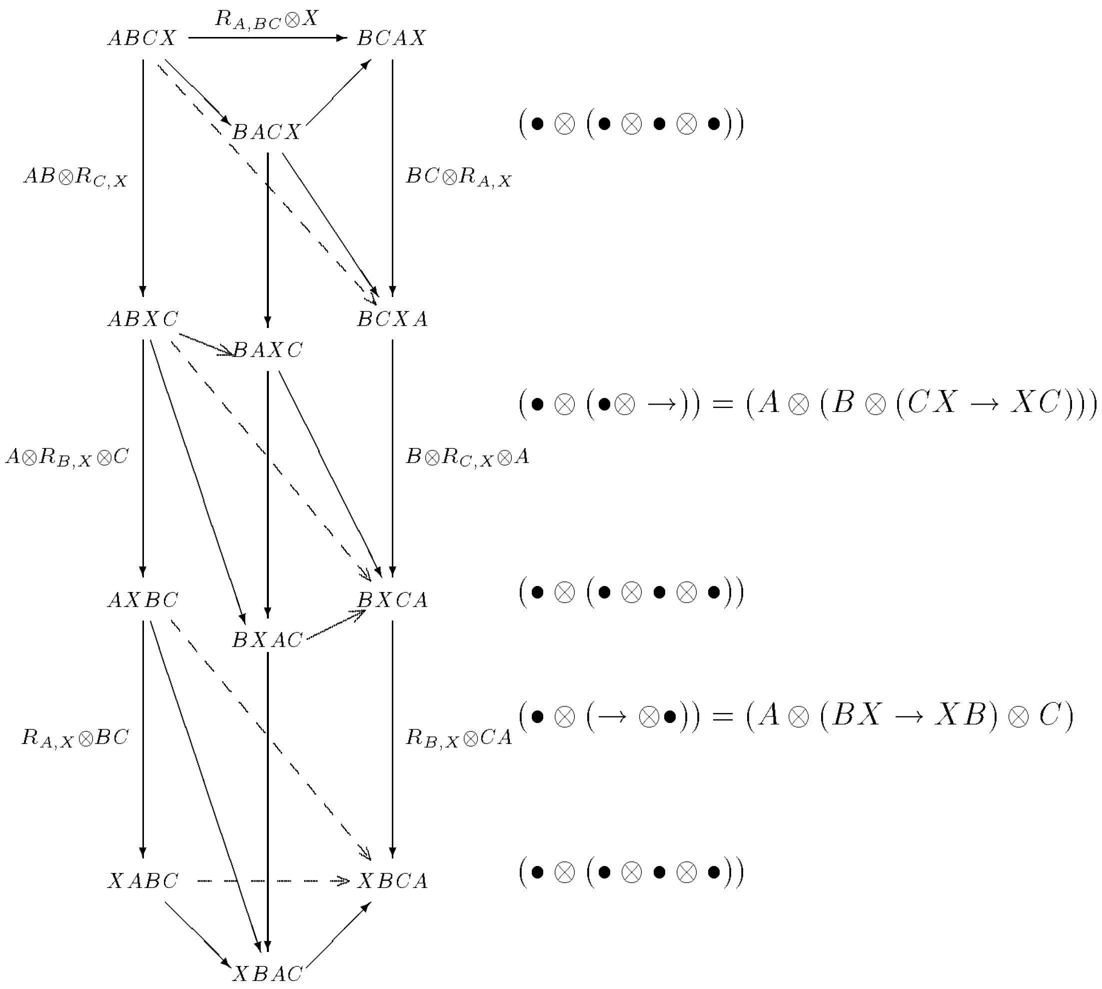
$((\bullet\otimes\bullet)\otimes\bullet)$: For any objects $(A, R_A, \tilde{R}_A), (B, R_B, \tilde{R}_B), (C, R_C, \tilde{R}_C) \in \mathcal{Z}(\mathcal{C})$ we have the 2-isomorphism $\tilde{R}_{(A,B|C)} := 1_{(R_A\otimes R_B)_C}$:

$$
\begin{tikzcd}
A\otimes B\otimes C \arrow[rr, "(R_A\otimes R_B)_C"] \arrow[dr, "A\otimes R_{B,C}"'] & \Uparrow 1 & C\otimes A\otimes B \\
& A\otimes C\otimes B \arrow[ur, "R_{A,C}\otimes B"'] &
\end{tikzcd}
$$

$((\bullet\otimes(\bullet\otimes\bullet))$: For any objects $(A, R_A, \tilde{R}_A), (B, R_B, \tilde{R}_B), (C, R_C, \tilde{R}_C) \in \mathcal{Z}(\mathcal{C})$ we have the 2-isomorphism $\tilde{R}_{(A|B,C)}$:

$$
\begin{tikzcd}
A\otimes B\otimes C \arrow[rr, "R_{A,(B\otimes C)}"] \arrow[dr, "R_{A,B}\otimes C"'] & \Uparrow \tilde{R}_{(A|B,C)} & B\otimes C\otimes A \\
& B\otimes A\otimes C \arrow[ur, "B\otimes R_{A,C}"'] &
\end{tikzcd}
$$

To verify that $\tilde{R}_{(A|-,-)}$ is a 2-morphism in $\mathcal{Z}(\mathcal{C})$, we have to check $(\Downarrow \otimes \bullet)$. The next diagram gives the proof.



The top triangle corresponds to the 2-morphism $\tilde{R}_{(A|B,C)} \otimes X$. The bottom triangle corresponds to the 2-morphism $X \otimes \tilde{R}_{(A|B,C)}$. The back side is $R_{R_{A,B\otimes C},X}$, the left front side is $R_{R_{A,B}\otimes C,X}$ and the right front side is $R_{B\otimes R_{A,C},X}$. The decomposition is indicated in the diagram.

Now we have to verify that these data satisfy all the axioms of a braided monoidal 2-category. The tetrahedron $(\bullet\otimes(\bullet\otimes\bullet\otimes\bullet))$ commutes by the definition of the objects of $\mathcal{Z}(\mathcal{C})$. The diagram $((\bullet\otimes\bullet)\otimes(\bullet\otimes\bullet))$ commutes by the definition of the tensor product of two objects in $\mathcal{Z}(\mathcal{C})$. By the same definition can be shown that $((\bullet \otimes \bullet \otimes \bullet) \otimes \bullet)$ commutes in $\mathcal{Z}(\mathcal{C})$. Note that because of our special choice of the 2-morphism $R_{R_{A,B},-)}$

that completes the morphism $R_{A,B}$ to a morphism in $\mathcal{Z}(\mathcal{C})$, the two 2-morphisms S^+ and S^- are equal in $\mathcal{Z}(\mathcal{C})$.

The other axioms of a braided monoidal 2-category are either part of our definitions, or else we have indicated within our Remarks which definitions imply them. We may summarize by stating:

Theorem 15. *Given any semistrict monoidal category $\mathcal{C}$, the center $\mathcal{Z}(\mathcal{C})$ is semistrict braided monoidal 2-category.*

4 Embedding $\mathcal{C}$ in $\mathcal{Z}(\mathcal{C})$

Given a semistrict braided monoidal 2-category $\mathcal{C}$, we would like to embed it in its center by a braided monoidal 2-functor $\mathcal{F} : \mathcal{C} \to \mathcal{Z}(\mathcal{C})$. Developing a general definition of 'braided monoidal 2-functor' would require a fair amount of work. Luckily, in our case we can restrict ourselves to a very strict sort of braided monoidal 2-functor which is easy to define. The following definition should not be taken as fundamental; it is simply designed to be the strictest one for which our embedding theorem holds.

Definition 16 *Let $(\mathcal{C}, \otimes, I, R, \tilde{R}_{(-|-,-)}, \tilde{R}_{(-,-|-)})$ and $(\mathcal{C}', \otimes', I, R', \tilde{R}'_{(-|-,-)}, \tilde{R}'_{(-,-|-)})$ be braided monoidal 2-categories. A monoidal 2-functor consists of:*

- *A 2-functor $\mathcal{F} : \mathcal{C} \to \mathcal{C}'$ such that $\mathcal{F}(I) = I'$.*
- *A pseudonatural transformation*

$$\xi : (\mathcal{F} \otimes_{\mathrm{G}} \mathcal{F}) \circ \otimes' \Rightarrow \otimes \circ \mathcal{F},$$

- *an invertible modification $\alpha : (1 \otimes \xi) \circ \xi \Rightarrow (\xi \otimes 1) \circ \xi$,*

such that the following diagram commutes.

$$
\begin{tikzcd}
 & F(XY)F(Z)F(W) \arrow[rr] \arrow[dd, dashed] \arrow[r, phantom, "2."] & & F(XYZ)F(W) \arrow[dd] \\
F(X)F(Y)F(Z)F(W) \arrow[ur] \arrow[rr] \arrow[dd] & & F(X)F(YZ)F(W) \arrow[ur, "5."] \arrow[dd, "4."] & \\
 & F(XY)F(ZW) \arrow[rr, dashed] \arrow[u, phantom, "1."] & & F(XYZW) \\
F(X)F(Y)F(ZW) \arrow[ur, dashed, "3."'] \arrow[rr] & & F(X)F(YZW) \arrow[ur] \arrow[u, phantom, "6."] &
\end{tikzcd}
$$

$1.\hat{=} \otimes_{\xi,\xi}$ $\quad 2.\hat{=} \alpha_{X\otimes Y,Z,W}$ $\quad 3.\hat{=} \alpha_{X,Y,Z\otimes W}$ $\quad 4.\hat{=} \alpha_{X,Y\otimes Z,W}$

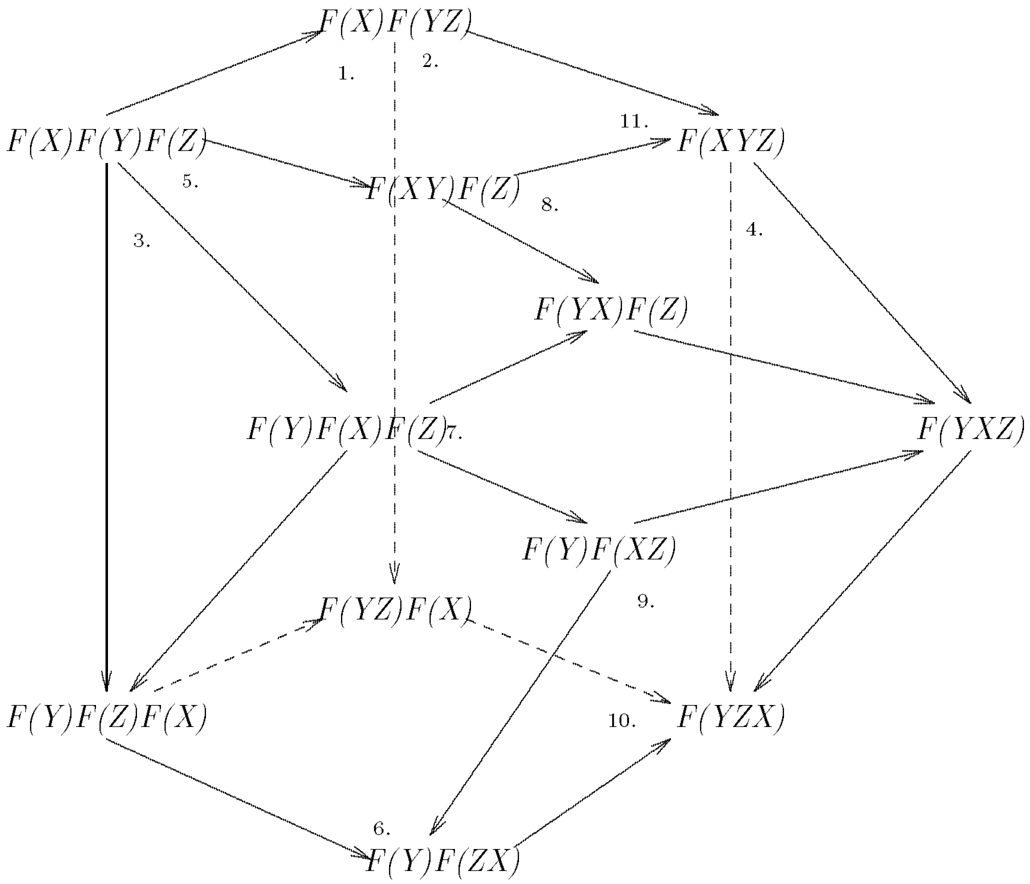
$5.\hat{=} \alpha_{X,Y,Z} \otimes \mathcal{F}(W)$ $\quad 6.\hat{=} \mathcal{F}(X) \otimes \alpha_{Y,Z,W}$

Definition 17 *A braided monoidal 2-functor consists of*

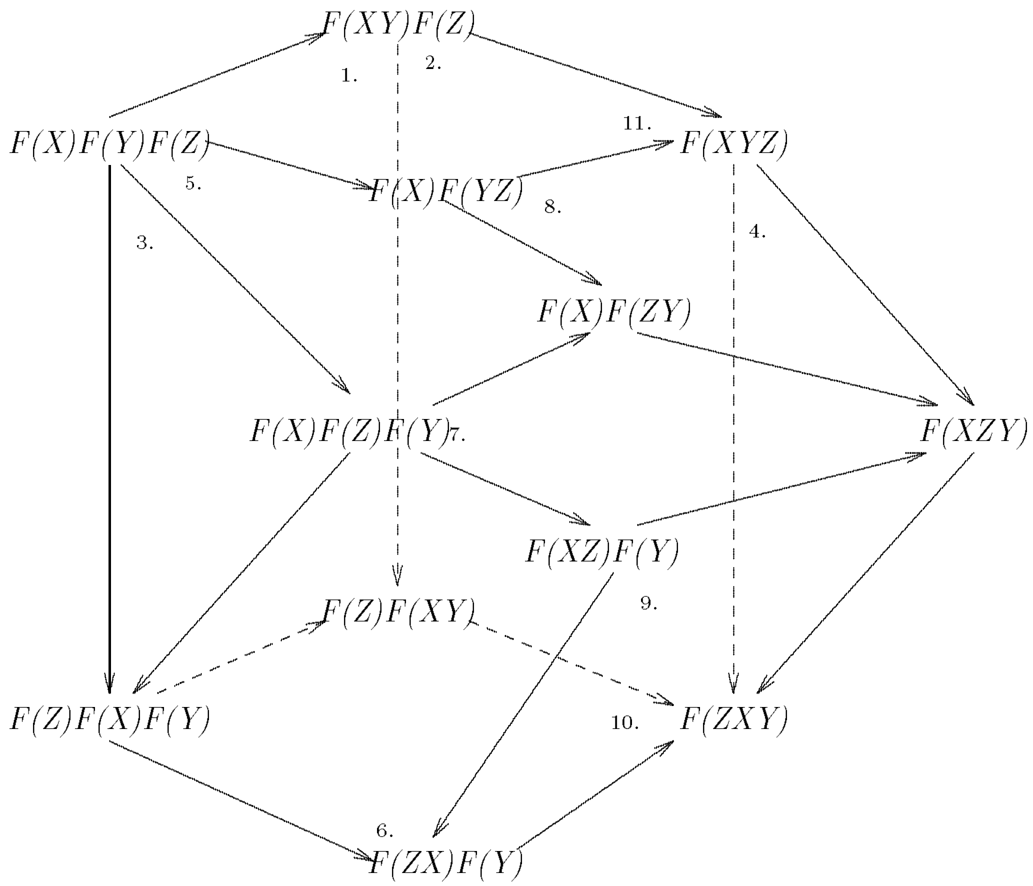
- *a monoidal 2-functor* $(\mathcal{F}, \xi, \alpha)$
- *a modification*

$$\mathcal{F}_R : \xi \circ \mathcal{F}(R) \Rightarrow R' \circ \xi,$$

such that the following two diagrams commute, expressing the fact that $\mathcal{F}$ *respects the modifications* $\tilde{R}_{(-|-,-)}$ *and* $\tilde{R}_{(-,-|-)}$ *up to* ξ*.*



$1.\hat{=}\ R'_{(\mathcal{F}(X),\xi)}$	$2.\hat{=}\ \mathcal{F}_R$	$3.\hat{=}\ \tilde{R}'_{(\mathcal{F}(X)\vert\mathcal{F}(Y),\mathcal{F}(Z))}$	$4.\hat{=}\ \mathcal{F}(\tilde{R}_{(X\vert Y,Z)})$
$5.\hat{=}\ \mathcal{F}_R \otimes \mathcal{F}(Z)$	$6.\hat{=}\ \mathcal{F}(Y) \otimes \mathcal{F}_R$	$7.\hat{=}\ \alpha_{Y,X,Z}$	$8.\hat{=}\ \xi_{R,Z}$
$9.\hat{=}\ \xi_{Y,R}$	$10.\hat{=}\ \alpha_{Y,Z,X}$	$11.\hat{=}\ \alpha_{X,Y,Z}$	



1.$\hat{=}$ $R'_{(\xi,\mathcal{F}(Z))}$	2.$\hat{=}$ $\mathcal{F}_R$	3.$\hat{=}$ $\tilde{R}'_{(\mathcal{F}(A),\mathcal{F}(B)\vert\mathcal{F}(Z))}$	4.$\hat{=}$ $\mathcal{F}(\tilde{R}_{(X,Y\vert Z)})$
5.$\hat{=}$ $\mathcal{F}(X)\otimes\mathcal{F}_R$	6.$\hat{=}$ $\mathcal{F}_R\otimes\mathcal{F}(Y)$	7.$\hat{=}$ $\alpha_{X,Z,Y}$	8.$\hat{=}$ $\xi_{X,R}$
9.$\hat{=}$ $\xi_{R,Y}$	10.$\hat{=}$ $\alpha_{Z,X,Y}$	11.$\hat{=}$ $\alpha_{X,Y,Z}$	

Theorem 18. *Let* $(\mathcal{C},\otimes,I,T,\tilde{T}_{(-\vert-,-)},\tilde{T}_{(-,-\vert-)})$ *be a semistrict braided monoidal 2-category, and let* $\mathcal{Z}(\mathcal{C})$ *be its center. Then there is a braided monoidal 2-functor* $\mathcal{F}:\mathcal{C}\to\mathcal{Z}(\mathcal{C})$ *given as follows:*

$$\begin{aligned}\mathcal{F}(A) &= (A, T_{A,-}, \tilde{T}_{(A\vert-,-)})\\ \mathcal{F}(f) &= (f, T_{f,-})\\ \mathcal{F}(\alpha) &= \alpha\end{aligned}$$

Moreover $\mathcal{F}$ *is injective on objects, morphisms and 2-morphisms, and surjective on 2-morphisms.*

Proof - First let us show that $\mathcal{F}$ is a monoidal 2-functor. For this, we must define a pseudonatural 1-morphism $\xi_{A,B} : \mathcal{F}(A) \otimes \mathcal{F}(B) \to \mathcal{F}(A \otimes B)$, where $A, B \in \mathcal{C}$. We let

$$\begin{aligned} \xi_{A,B} &:= (1_{A\otimes B}, \tilde{T}^{-1}_{(A,B|\text{-})}) : \\ &\qquad (A, T_{A,\text{-}}, \tilde{T}_{(A|\text{-},\text{-})}) \otimes_{\mathcal{Z}(\mathcal{C})} (B, T_{B,\text{-}}, \tilde{T}_{(B|\text{-},\text{-})}) \to (A \otimes B, T_{A\otimes B,\text{-}}, \tilde{T}_{(A\otimes B|\text{-},\text{-})}) \end{aligned}$$

To be a morphism in $\mathcal{Z}(\mathcal{C})$, $\xi_{A,B}$ has to satisfy $(\to \otimes(\bullet \otimes \bullet))$. This is equivalent to the axiom $((\bullet \otimes \bullet) \otimes (\bullet \otimes \bullet))$ in $\mathcal{C}$. We show that ξ is natural, not merely pseudonatural. To this end we first show that for any morphism $f \colon A \to A'$ in $\mathcal{C}$ the following diagram commutes 'on the nose'. (Remember our shorthand symbol for tensor products in $\mathcal{Z}(\mathcal{C})$.)

$$\begin{tikzcd}
(A \otimes B, T_A \otimes T_B, \tilde{T}_A \otimes \tilde{T}_B) \arrow[r, "{(1_{A\otimes B}, \tilde{T}^{-1}_{(A,B|\text{-})})}"] \arrow[d, "{(f, T_{f,\text{-}}) \otimes_{\mathcal{Z}(\mathcal{C})} B}"'] & (A \otimes B, T_{A\otimes B,\text{-}}, \tilde{T}_{(A\otimes B|\text{-},\text{-})}) \arrow[d, "{(f \otimes B, T_{f\otimes B,\text{-}})}"] \\
(A' \otimes B, T_{A'} \otimes T_B . \tilde{T}_{A'} \otimes \tilde{T}_B) \arrow[r, "{(1_{A'\otimes B}, \tilde{T}_{(A',B|\text{-})})}"'] & (A' \otimes B, T_{A'\otimes B,\text{-}}, \tilde{T}_{(A'\otimes B|\text{-},\text{-})})
\end{tikzcd}$$

The morphism "first right, then down" equals

$$(f \otimes B, T_{(f\otimes B,\text{-})} \cdot (\tilde{T}^{-1}_{(A,B|\text{-})} \circ (\text{-} \otimes f \otimes B)))$$

The morphism "first down, then right" equals

$$(f \otimes B, ((f \otimes B \otimes \text{-}) \circ \tilde{T}^{-1}_{(A,B|\text{-})}) \cdot (\otimes_{f,T_{(B,X)}} \circ (T_{A',X} \otimes B)) \cdot ((A \otimes T_{B,X}) \circ (T_{f,X} \otimes B)))$$

These two $\mathcal{Z}(\mathcal{C})$-morphisms are equal, since by $((\to \otimes \bullet) \otimes \bullet)$, the underlying 2-morphisms are equal:

$$((\to \otimes \bullet) \otimes \bullet) \qquad \begin{tikzcd}
ABX \arrow[rr, "T_{AB,X}"] \arrow[dr] \arrow[dd, "f\otimes BX"'] \arrow[dr, phantom, shift left=3, "\Uparrow \tilde{T}_{(A,B|X)}"] & \arrow[d, phantom, "\Uparrow T_{f\otimes B,X}"] & XAB \arrow[dd, "X\otimes f\otimes B"] \\
\arrow[r, phantom, "\Uparrow \otimes_{(f,T_{B,X}}"] & AXB \arrow[ur] \arrow[dd] \arrow[r, phantom, "\Uparrow T_{f,X} B"] & \\
A'BX \arrow[rr, dashed] \arrow[dr] \arrow[dr, phantom, shift left=3, "\Uparrow \tilde{T}_{(A',B|X)}"] & & XA'B \\
& A'XB \arrow[ur] &
\end{tikzcd}$$

Then we must show naturality with respect to morphisms of the form $g: B \to B'$, which is similar. Finally, it is easy to show that ξ is also compatible with 2-morphisms. Using the axiom $((\bullet \otimes \bullet \otimes \bullet) \otimes \bullet)$ we see that ξ fullfills the associativity condition on the nose, so we can define α to be the identity.

Next, we show that $\mathcal{F}$ is braided and, in addition, $\mathcal{F}_R = \mathrm{id}$.

$$\mathcal{F}_R = \mathrm{id}: \xi \circ \mathcal{F}(T) = R^{\mathcal{Z}(\mathcal{C})} \circ \xi.$$

This is done by the following calculation.

$$\begin{aligned}
(\xi \circ \mathcal{F}(T) \circ \xi^{-1})_{A,B} &= (1_{A\otimes B}, \tilde{T}^{-1}_{(A,B|\text{-})}) \circ (T_{A,B}, T_{T_{A,B},\text{-}}) \circ (1_{A\otimes B}, \tilde{T}_{(A,B|\text{-})}) \\
&= (T_{A,B}, S^{-}_{A,B,\text{-}}) \\
&= (T_{A,B}, S^{+}_{A,B,\text{-}}) \\
&= (T_{A,B}, R_{(T_{A,B},\text{-})}) \\
&= R^{\mathcal{Z}(\mathcal{C})}_{A,B}
\end{aligned}$$

Here $\xi^{-1} = (1_{A\otimes B}, \tilde{T}_{(A,B|\text{-})})$ is the inverse of ξ, as can be easily verified using the composition law for 1-morphisms in $\mathcal{Z}(\mathcal{C})$. The third equation holds by our assumption that $S^+ = S^-$. The fifth equation holds according to our definition of the braiding in $\mathcal{Z}(\mathcal{C})$.

Finally, we must check that both diagrams in the definition of a strong braided monoidal 2-functor commute. In the first diagram all the 2-morphisms except 3 and 4 are identities. Note that the 2-morphism labeled 1, namely $T_{(\mathcal{F}(X),\xi)}$, is the identity, since the 1-morphism part of ξ is the identity, and that by an application of axiom $(\bullet\otimes \Downarrow)$, face 1 commutes on the nose. The remaining 2-morphisms 3 and 4 are equal.

In the second diagram the 2-morphism 3 is the identity. Here, the 2-morphism 1 is defined to be $R^{-1}_{(X,Y|Z)}$ and hence agrees with 4. The remaining 2-morphisms are identities, so this diagram also commutes. □

5 Conclusions

While we have made some progress in understanding monoidal 2-categories and braided monoidal 2-categories, it seems clear that a truly elegant, not to mention correct, treatment of these concepts requires a better understanding of 3-categories and 4-categories. In this spirit, we would like to conclude with a list of some issues that are not yet resolved.

1) We have not included in our definition of semistrict braided monoidal 2-category any axioms involving the unit object I (other than those appearing in the definition of semistrict monoidal 2-category). In the case of a strict braided monoidal category, where $A \otimes I = A \otimes I = A$ for any object A, there are theorems saying that $R_{A,I} = R_{I,A} = 1_A$. In the 2-categorical setting the proof for this theorem turns into an isomorphism: $R_{A,I} \cong R_{I,A} \cong 1_A$. If we assumed these isomorphisms were equations and in addition that $R_{f,I} = R_{I,f} = 1_f$ for all 1-morphisms $f : I \to I$, then we could conclude that any braided monoidal category with one object gives a symmetric monoidal category, as expected.

2) In the center $\mathcal{Z}(\mathcal{C})$ as we have defined it, the 2-morphisms $\tilde{R}_{(A,B|C)}$ are all identity 2-morphisms, while the 2-morphisms $\tilde{R}_{(A|B,C)}$ are not. This points to a curious assymetry in our definition of the center. One could equally well have defined the center so that $\tilde{R}_{(A|B,C)}$ was always the identity, and not $\tilde{R}_{(A,B|C)}$, but the question is: why is any 'symmetry-breaking' required? It may be relevant that Gordon, Power and Street's proof [17] that any tricategory is triequivalent to a semistrict 3-category involves a 'symmetry-breaking' maneuver. This occurs because the definition of semistrict 3-category has an inherent asymmetry, in that given $f\colon A \to A'$ and $g\colon B \to B'$, the 2-morphism $\otimes_{f,g}$ goes from $(A \otimes g)(f \otimes B')$ to $(f \otimes B)(A' \otimes g)$ rather than vice versa. Perhaps, therefore, the center construction involves no asymmetries at the level of weak n-categories, but an arbitrary symmetry breaking is needed to translate it into the framework of semistrict n-categories.

Because the strong braided monoidal 2-functor of Theorem 18 is injective on objects, morphisms and 2-morphisms, this result thus serves as a strictification theorem asserting that any semistrict braided monoidal 2-category $\mathcal{C}$ is equivalent (in a precise sense) to one for which $\tilde{R}_{(-,-|-)}$ is trivial. Indeed, one may prove this strictification in other ways as well. One can also, of course, show that any any semistrict braided monoidal 2-category $\mathcal{C}$ is equivalent in the same sense to one for which $\tilde{R}_{(-|-,-)}$ is trivial.

Acknowledgements

Many of the basic ideas behind this paper were developed in collaboration with James Dolan. We would also like to thank Lawrence Breen, Misha Kapranov, John Power, James Stasheff and Ross Street for useful discussions and correspondence. J.B. is grateful to the Mathematical Institute of the University of Munich for their hospitality while some of this work was done.

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