# Categorified Gauge Theory

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joint work with:
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# in honor of Larry Breen's 60th birthday

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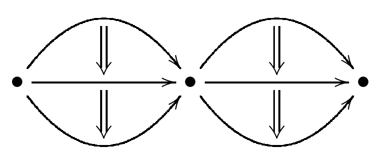

More details at:

**http://math.ucr.edu/home/baez/breen/**

## Gauge Theory

Ordinary gauge theory describes how 0-dimensional particles transform as we move them along 1-dimensional paths. It is natural to assign a Lie group element to each path:

$$\bullet \xrightarrow{g} \bullet$$

since composition of paths then corresponds to multiplication:

$$\bullet \xrightarrow{g} \bullet \xrightarrow{g'} \bullet$$

while reversing the direction of a path corresponds to taking inverses:

$$\bullet \xleftarrow{g^{-1}} \bullet$$

and the associative law makes the holonomy along a triple composite unambiguous:

$$\bullet \xrightarrow{g} \bullet \xrightarrow{g'} \bullet \xrightarrow{g''} \bullet$$

In short: *the topology dictates the algebra!*

The electromagnetic field is described by a connection where the group is U(1). Other forces are described using other groups.

To *really* let the topology dictate the algebra, we should replace the Lie group by a 'smooth groupoid': a groupoid in some convenient category of smooth spaces. Mackaay and Picken have noted that for any manifold $M$ there is a smooth groupoid $\mathcal{P}_1(M)$, the **path groupoid**, for which:

- objects are points $x \in M$,
- morphisms are thin homotopy classes of smooth paths $\gamma\colon [0,1] \to M$ such that $\gamma(t)$ is constant near $t = 0, 1$.

For any Lie group $G$, a principal $G$-bundle $P \to M$ gives a smooth groupoid $\mathrm{Trans}(P)$, the **transport groupoid**, for which:

- objects are torsors $P_x$ for $x \in M$,
- morphisms are torsor morphisms.

Via parallel transport, any connection on $P$ gives a smooth functor called its **holonomy**:

$$\mathrm{hol}\colon \mathcal{P}_1(M) \to \mathrm{Trans}(P)$$

A trivialization of $P$ makes $\mathrm{Trans}(P)$ equivalent to $G$, so it gives:

$$\mathrm{hol}\colon \mathcal{P}_1(M) \to G$$

Next let's study how 1-dimensional 'strings' transform as we move them along 2-dimensional surfaces. Naively we might wish our holonomy to assign a group element to each surface like this:

$$\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow g"] & \bullet \end{tikzcd}$$

We can compose surfaces of this sort vertically:

$$\begin{tikzcd} \bullet \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=3.5ex, "\Downarrow g"] \arrow[r, phantom, shift right=3.5ex, "\Downarrow g'"] & \bullet \end{tikzcd}$$

and horizontally:

$$\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow g"] & \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow g'"] & \bullet \end{tikzcd}$$

Suppose both of these correspond to multiplication in some Lie group $G$. To obtain a well-defined holonomy for this surface regardless of whether we do vertical or horizontal composition first:

$$\begin{tikzcd} \bullet \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=3.5ex, "\Downarrow g_1"] \arrow[r, phantom, shift right=3.5ex, "\Downarrow g_2"] & \bullet \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=3.5ex, "\Downarrow g_1'"] \arrow[r, phantom, shift right=3.5ex, "\Downarrow g_2'"] & \bullet \end{tikzcd}$$

we must have

$$(g_1 g_2)(g_1' g_2') = (g_1 g_1')(g_2 g_2').$$

This forces $G$ to be abelian!

Pursuing this approach, we ultimately get the theory of connections on 'abelian gerbes'. If $G = \mathrm{U}(1)$, such a connection looks locally like a 2-form — and it shows up naturally in string theory, satisfying equations very much like those of electromagnetism!

To go beyond this and get *nonabelian* higher gauge fields, we should let the topology dictate the algebra, and consider a connection that gives holonomies *both for paths and for surfaces.*

So, let's replace the path groupoid by some 2-groupoid where:

- objects are points of $M$: $\bullet\, x$
- morphisms are certain paths in $M$: $\bullet \xrightarrow{\gamma} \bullet$
- 2-morphisms are certain equivalence classes of paths of paths in $M$:
$$\begin{tikzcd}[column sep=large] \bullet \arrow[r, bend left=60, ""{name=U, below}] \arrow[r, bend right=60, ""{name=D, above}] & \bullet \arrow[Rightarrow, from=U, to=D, "f"] \end{tikzcd}$$

A 2-groupoid allows composition of morphisms:

$$\bullet \xrightarrow{\gamma} \bullet \xrightarrow{\gamma'} \bullet$$

vertical composition of 2-morphisms:

$$\begin{tikzcd}[column sep=large] \bullet \arrow[r, bend left=60, ""{name=U, below}] \arrow[r, ""{name=M, above}, ""{name=N, below}] \arrow[r, bend right=60, ""{name=D, above}] & \bullet \arrow[Rightarrow, from=U, to=M, "f"] \arrow[Rightarrow, from=N, to=D, "g"] \end{tikzcd}$$

and horizontal composition of morphisms:

$$\begin{tikzcd}[column sep=large] \bullet \arrow[r, bend left, ""{name=U, below}] \arrow[r, bend right, ""{name=D, above}] & \bullet \arrow[r, bend left, ""{name=U2, below}] \arrow[r, bend right, ""{name=D2, above}] & \bullet \arrow[Rightarrow, from=U, to=D, "f"] \arrow[Rightarrow, from=U2, to=D2, "f'"] \end{tikzcd}$$

satisfying various laws, including one that makes this unambiguous:

$$\begin{tikzcd}[column sep=large] \bullet \arrow[r, bend left=60, ""{name=U, below}] \arrow[r, ""{name=M, above}, ""{name=N, below}] \arrow[r, bend right=60, ""{name=D, above}] & \bullet \arrow[r, bend left=60, ""{name=U2, below}] \arrow[r, ""{name=M2, above}, ""{name=N2, below}] \arrow[r, bend right=60, ""{name=D2, above}] & \bullet \arrow[Rightarrow, from=U, to=M, "f"] \arrow[Rightarrow, from=N, to=D, "g"] \arrow[Rightarrow, from=U2, to=M2, "f'"] \arrow[Rightarrow, from=N2, to=D2, "g'"] \end{tikzcd}$$

More precisely, define the **path 2-groupoid** $\mathcal{P}_2(M)$ to be the smooth 2-groupoid in which:

- objects are points $x \in M$,
- morphisms are smooth paths $\gamma\colon [0,1] \to M$ with $\gamma(t)$ constant near $t = 0, 1$,
- 2-morphisms are thin homotopy classes of smooth maps $f\colon [0,1]^2 \to M$ with $f(s,t)$ independent of $s$ near $s = 0, 1$ and constant near $t = 0, 1$.

We might hope for something like this:

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For any Lie 2-group $\mathcal{G}$, a principal $\mathcal{G}$-2-bundle $P \to M$ gives a smooth 2-groupoid $\mathrm{Trans}(P)$ where:

- objects are 2-torsors $P_x$,
- morphisms are 2-torsor morphisms, $f\colon P_x \to P_y$
- 2-morphisms are 2-torsor 2-morphisms $\theta\colon f \Rightarrow g$.

Via parallel transport, a 2-connection on $P$ gives a smooth 2-functor called its **holonomy**:

$$\mathrm{hol}\colon \mathcal{P}_2(M) \to \mathrm{Trans}(P).$$

A trivialization of $P$ makes $\mathrm{Trans}(P)$ equivalent to $\mathcal{G}$ so it gives

$$\mathrm{hol}\colon \mathcal{P}_2(M) \to \mathcal{G}.$$

??????????????????????????????????????????????????

*Can we make this precise? Is it true?*

## Internalization

The crucial trick is 'internalization'. Ehresmann and Lawvere showed how to 'internalize' concepts by defining them in terms of commutative diagrams:

A **small category**, say $C$, has a <u>set</u> of objects $\mathrm{Ob}(C)$, a <u>set</u> of morphisms $\mathrm{Mor}(C)$, source and target <u>functions</u>

$$s, t \colon \mathrm{Ob}(C) \to \mathrm{Mor}(C),$$

a composition <u>function</u>

$$\circ \colon \mathrm{Mor}(C)_s{\times}_t\mathrm{Mor}(C) \to \mathrm{Mor}(C)$$

and an identity–assigning <u>function</u>

$$\mathrm{id} \colon \mathrm{Ob}(C) \to \mathrm{Mor}(C)$$

making these diagrams commute. . . .

and letting these diagrams live within some category $K$:

A **category in** $K$, say $C$, has an <u>object</u> $\mathrm{Ob}(C) \in K$, an <u>object</u> $\mathrm{Mor}(C) \in K$, source and target <u>morphisms</u>

$$s, t \colon \mathrm{Ob}(C) \to \mathrm{Mor}(C),$$

a composition <u>morphism</u>

$$\circ \colon \mathrm{Mor}(C)_s{\times}_t\mathrm{Mor}(C) \to \mathrm{Mor}(C)$$

and an identity-assigning <u>morphism</u>

$$\mathrm{id} \colon \mathrm{Ob}(C) \to \mathrm{Mor}(C)$$

making these diagrams commute. . . .

Similarly we can define **functors in** $K$ and **natural transformations in** $K$, obtaining a 2-category $\boldsymbol{K}\mathbf{Cat}$. We can also define **groups in** $K$ and **homomorphisms in** $K$, obtaining a category $\boldsymbol{K}\mathbf{Grp}$.

## Smooth Categories, 2-Groups, and Lie 2-Groups

We can categorify concepts from differential geometry with the help of internalization:

- A **smooth category** is a category in Diff.
- A **strict 2-group** (or **categorical group**) is a category in Grp.
- A **strict Lie 2-group** is a category in LieGrp.

A strict 2-group is the same as a strict monoidal category such that:

- for every object $x$ there exists an object $y$ with $x \otimes y = 1$, $y \otimes x = 1$;
- for every morphism $f$ there exists a morphism $g$ with $fg = 1$, $gf = 1$.

More generally, a **2-group** (or **gr-category**) is a weak monoidal category such that:

- for every object $x$ there is a specified object $x^{-1}$ equipped with isomorphisms

  $$i_x \colon 1 \to x \otimes x^{-1}, \quad e_x \colon x^{-1} \otimes x \to 1$$

  forming an adjunction;
- for every morphism $f$ there exists a morphism $g$ with $fg = 1$, $gf = 1$.

We can also define general **Lie 2-groups** the same way, working in DiffCat rather than Cat.

## Examples of 2-Groups

1) Any abelian group $A$ gives a strict 2-group with one object and $A$ as the automorphisms of this object. Lie 2-groups of this kind will be structure 2-groups of 2-bundles having an *abelian gerbe* of sections.

2) Any category $C$ gives a 2-group $\mathrm{Aut}(C)$ whose objects are equivalences $f\colon C \to C$ and whose morphisms are natural isomorphisms between these.

3) A group $H$ is a category with one object and all morphisms invertible. In this case, 2) gives a strict 2-group $\mathrm{Aut}(H)$ whose objects are automorphisms of $H$ and whose morphisms from $f$ to $f'$ are elements $k \in H$ with $f'(h) = kf(h)k^{-1}$.

4) Any Lie group $H$ gives a strict Lie 2-group $\mathrm{Aut}(H)$ defined as in 3) but with everything smooth. Lie 2-groups of this sort will be structure 2-groups of 2-bundles having a *nonabelian gerbe* of sections.

. . . and many 'more concrete' examples, some listed in my paper with Aaron Lauda.

## 2-Bundles

Toby Bartels has developed a theory of '2-bundles'. We can think of a manifold $M$ as a smooth category with only identity morphisms. A **2-bundle** over $M$ consists of:

- a smooth category $P$ (the **total space**),
- a smooth category $F$ (the **standard fiber**),
- a smooth functor $p\colon P \to M$ (the **projection**),

such that each point $x \in M$ has an open neighborhood $U$ for which there exists a smooth equivalence:

$$f\colon p^{-1}U \to U \times F$$

such that this diagram commutes:

$$\begin{tikzcd} p^{-1}U \arrow[rr, "f"] \arrow[dr, "p|_{p^{-1}U}"'] & & U \times F \arrow[dl] \\ & U & \end{tikzcd}$$

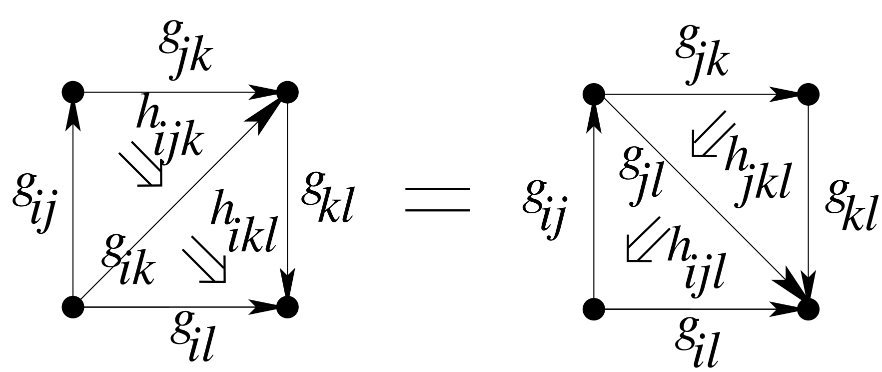
The equivalence $f$ is called a **local trivialization**.

If $F$ is a smooth category, $\mathcal{G} = \mathrm{Aut}(F)$ is a smooth 2-group. Given a 2-bundle $P \to M$ with standard fiber $F$, and choosing local trivializations over open sets $U_i$ covering $M$, we obtain:

- smooth maps
$$g_{ij} \colon U_i \cap U_j \to \mathrm{Ob}(\mathcal{G})$$
- smooth maps
$$h_{ijk} \colon U_i \cap U_j \cap U_k \to \mathrm{Mor}(\mathcal{G})$$
with
$$h_{ijk}(x) \colon g_{ij}(x) g_{jk}(x) \to g_{ik}(x)$$
- smooth maps
$$k_i \colon U_i \to \mathrm{Mor}(\mathcal{G})$$
with
$$k_i(x) \colon g_{ii}(x) \to 1 \in \mathcal{G}.$$

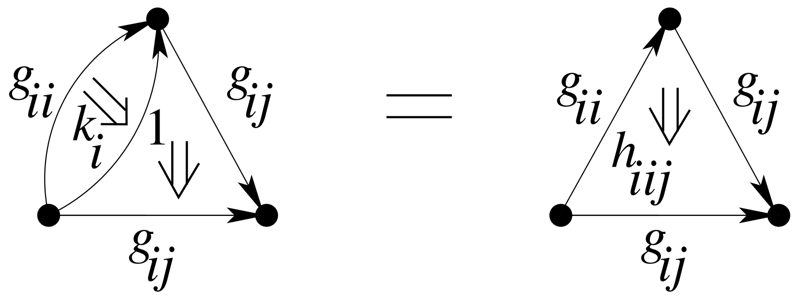
Furthermore:

- $h$ satisfies an equation on quadruple intersections $U_i \cap U_j \cap U_k \cap U_\ell$:

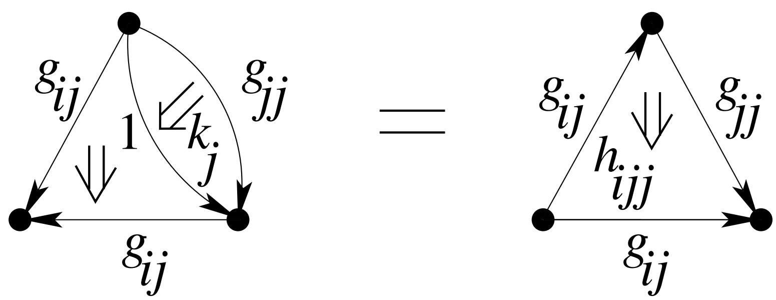



(the **associative law**)

- $k$ satisfies two equations on double intersections $U_i \cap U_j$:




(the **left unit law**) and




(the **right unit law**).

More generally, for any smooth 2-group $\mathcal{G}$ we say a 2-bundle $P \to M$ has $\mathcal{G}$ as its **structure 2-group** when $g_{ij}$, $h_{ijk}$, $k_i$ factor through an action $\mathcal{G} \to \mathrm{Aut}(F)$.

In particular, if $\mathcal{G}$ acts on $F = \mathcal{G}$ by left multiplication, $P$ is a **principal $\mathcal{G}$-2-bundle**. Its fibers are then $\mathcal{G}$-2-torsors in a suitable sense.

Any 2-bundle has a stack of sections. A principal $\mathcal{G}$-2-bundle with $\mathcal{G} = \mathrm{Aut}(H)$ for some Lie group $H$ has a nonabelian gerbe of sections!

## 2-Connections on Principal 2-Bundles

So far Urs Schreiber and I have only handled 2-connections on principal 2-bundles where the structure 2-group $\mathcal{G}$ is *strict*.

A strict Lie 2-group $\mathcal{G}$ is determined by:

- the Lie group $G$ consisting of all objects of $\mathcal{G}$,
- the Lie group $H$ consisting of all morphisms of $\mathcal{G}$ with source 1,
- the homomorphism $t\colon H \to G$ sending each morphism in $H$ to its target,
- the action $\alpha$ of $G$ on $H$ defined using conjugation in $\mathrm{Mor}(\mathcal{G})$ via
$$\alpha(g)h = 1_g h {1_g}^{-1}$$

The system $(G, H, t, \alpha)$ satisfies equations making it a **crossed module**. Conversely, any crossed module of Lie groups gives a strict Lie 2-group.

Let $\mathcal{G}$ be a strict Lie 2-group, let $(G, H, t, \alpha)$ be its crossed module, and let $(\mathfrak{g}, \mathfrak{h}, dt, d\alpha)$ be the corresponding **differential crossed module**.

If $P \to M$ is a principal 2-bundle with structure group $\mathcal{G}$ built using a cover $U_i$ of $M$, we can describe a **2-connection** on $P$ in terms of:

- a $\mathfrak{g}$-valued 1-form $A_i$ on each open set $U_i$,
- an $\mathfrak{h}$-valued 2-form $B_i$ on each open set $U_i$,

together with some extra data and equations for double and triple intersections — following the ideas of Breen and Messing.

If $P$ is trivial all this reduces to:

- a $\mathfrak{g}$-valued 1-form $A$ on $M$,
- an $\mathfrak{h}$-valued 2-form $B$ on $M$.

Let's restrict attention to this case and ponder the existence of a *holonomy* 2-functor

$$F\colon \mathcal{P}_2(M) \to \mathcal{G}$$

built using parallel transport.

## Parallel Transport

Recall: $\mathcal{G}$ is a strict Lie 2-group with crossed module $(G, H, t, \alpha)$. A 2-connection on a trivial principal $\mathcal{G}$-2-bundle over $M$ consists of:

- a $\mathfrak{g}$-valued 1-form $A$ on $M$,
- an $\mathfrak{h}$-valued 2-form $B$ on $M$.

This data determines a smooth **holonomy** 2-functor

$$\mathrm{hol}\colon \mathcal{P}_2(M) \to \mathcal{G}$$

if and only if the **fake curvature** vanishes:

$$F_A - dt(B) = 0,$$

where $F_A$ is the usual curvature of $A$, namely the $\mathfrak{g}$-valued 2-form

$$F_A = dA + A \wedge A.$$

The fake curvature vanishing ensures that parallel transport along a path of paths is *invariant under thin homotopies* — in particular, invariant under reparametrization! This implies that $\mathrm{hol}(f)$ is well-defined for any 2-morphism $f\colon \gamma \to \gamma'$ in the the path 2-groupoid.

Vanishing fake curvature is also needed to obtain

$$\mathrm{hol}(f)\colon \mathrm{hol}(\gamma) \to \mathrm{hol}(\gamma').$$

All this generalizes to nontrivial principal $\mathcal{G}$-2-bundles: we obtain a holonomy 2-functor

$$\mathrm{hol}\colon \mathcal{P}_2(M) \to \mathrm{Trans}(P)$$

if and only if the fake curvature vanishes.