Higher Gauge Theory and the String Group

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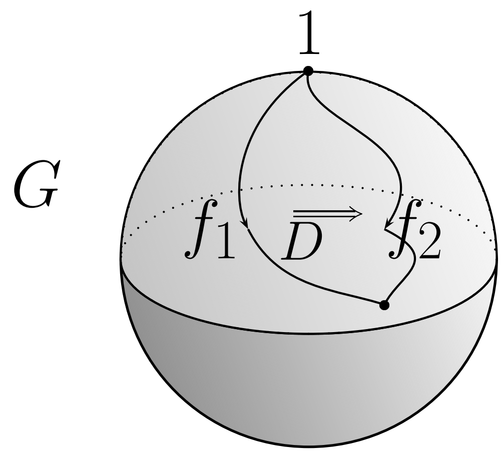


For more see:
http://math.ucr.edu/home/baez/btm22/

Categorification

sets $\rightsquigarrow$ **categories**
functions $\rightsquigarrow$ **functors**
equations $\rightsquigarrow$ **natural isomorphisms**

Categorification 'boosts the dimension' by one:

$$\begin{tikzcd} \bullet \arrow[r, bend left] & \bullet & \rightsquigarrow & \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \bullet \end{tikzcd}$$

In **strict** categorification we keep equations as equations. This is evil... but today we'll do it whenever it doesn't cause trouble, just to save time.

Higher Gauge Theory

groups $\rightsquigarrow$ **2-groups**
Lie algebras $\rightsquigarrow$ **Lie 2-algebras**
bundles $\rightsquigarrow$ **2-bundles**
connections $\rightsquigarrow$ **2-connections**

Connections describe parallel transport for particles.
2-Connections describe parallel transport for strings!

$$\begin{tikzcd} \bullet \arrow[r, bend left] & \bullet \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \bullet \arrow[r, bend left, ""{name=U, below}] \arrow[r, bend right, ""{name=D}] & \bullet \arrow[Rightarrow, from=U, to=D] \end{tikzcd}$$

We should even go beyond $n = 2$... but not today.

Fix a simply-connected compact simple Lie group G. Then:

- The Lie algebra $\mathfrak{g}$ gives a 1-parameter family of Lie 2-algebras $\mathfrak{string}_k(\mathfrak{g})$.
- When $k \in \mathbb{Z}$, $\mathfrak{string}_k(\mathfrak{g})$ comes from a Lie 2-group $\mathrm{String}_k(G)$.
- The 'geometric realization of the nerve' of $\mathrm{String}_k(G)$ is a topological group, $|\mathrm{String}_k(G)|$.
- Principal $\mathrm{String}_k(G)$-2-bundles are the same as $|\mathrm{String}_k(G)|$-bundles.
- For $k = 1$, $|\mathrm{String}_k(G)|$ is G with its 3rd homotopy group made trivial.
- We can define connections and characteristic classes for $\mathrm{String}_k(G)$-2-bundles!

2-Groups

A **strict 2-group** is a category in Grp: a category with a group of objects and a group of morphisms, such that all the category operations are group homomorphisms.

The objects in a 2-group look like this:

$$\begin{tikzcd} \bullet \arrow[r, bend left, "g"] & \bullet \end{tikzcd}$$

The morphisms look like this:

$$\begin{tikzcd} \bullet \arrow[r, bend left=50, "g"{name=U}] \arrow[r, bend right=50, "g'"'{name=D}] & \bullet \arrow[Rightarrow, from=U, to=D, "f"] \end{tikzcd}$$

We can multiply objects: $\bullet \xrightarrow{g} \bullet \xrightarrow{g'} \bullet$

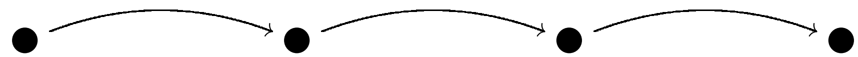
multiply morphisms:

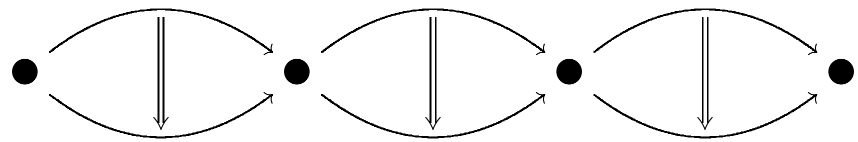
$$\begin{tikzcd}
\bullet \arrow[r, bend left=50, "g_1"] \arrow[r, bend right=50, "g_1'"'] \arrow[r, phantom, "f_1 \Downarrow"] & \bullet \arrow[r, bend left=50, "g_2"] \arrow[r, bend right=50, "g_2'"'] \arrow[r, phantom, "f_2 \Downarrow"] & \bullet
\end{tikzcd}$$

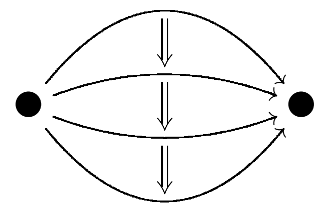
and compose morphisms:

$$\begin{tikzcd}
\bullet \arrow[r, bend left=60, "g"] \arrow[r, "g'" near start] \arrow[r, bend right=60, "g''"'] \arrow[r, phantom, shift left=2.5ex, "\Downarrow f"] \arrow[r, phantom, shift right=2.5ex, "\Downarrow f'"] & \bullet
\end{tikzcd}$$

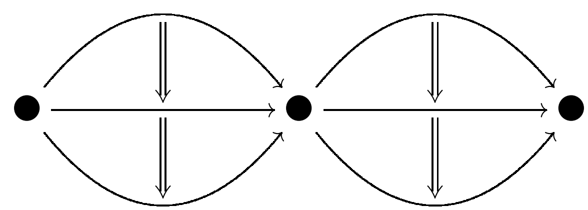
All 3 operations have a unit and inverses. All 3 are associative, so these are well-defined:







Finally, the **interchange law** holds, meaning



is well-defined.

Mac Lane and Whitehead first introduced 2-groups in the disguise of 'crossed modules':

$$G_0 \xleftarrow{\partial} G_1$$

Here G_0 and G_1 are groups, and G_0 acts on G_1 in a manner compatible with the differential ∂.

To get a crossed module from a 2-group, just let G_0 be the group of objects:

$$\begin{tikzcd} \bullet \arrow[r, bend left, "g"] & \bullet \end{tikzcd}$$

and G_1 be the group of morphisms starting at 1. The differential ∂ is defined as follows:

$$\begin{tikzcd} \bullet \arrow[r, bend left=50, "1"] \arrow[r, bend right=50, "\partial(f)"'] \arrow[r, phantom, "\Downarrow f"] & \bullet \end{tikzcd}$$

Lie 2-Algebras

A **strict Lie 2-algebra** is a category in LieAlg: a category with a Lie algebra of objects and a Lie algebra of morphisms, such that all the category operations are Lie algebra homomorphisms.

A strict Lie 2-algebra can be viewed as an 'infinitesimal crossed module':

$$\mathfrak{g}_0 \xleftarrow{\partial} \mathfrak{g}_1$$

Here $\mathfrak{g}_0$ and $\mathfrak{g}_1$ are Lie algebras, and $\mathfrak{g}_0$ acts as derivations of $\mathfrak{g}_1$ in a manner compatible with the differential ∂.

Theorem (Mac Lane, Sinh). A 2-group is determined up to equivalence by:

- the group G of isomorphism classes of objects,
- the abelian group A of endomorphisms of any object,
- an action of G on A,
- an element of $H^3(G, A)$.

Theorem (Gerstenhaber, Crans). A Lie 2-algebra is determined up to equivalence by:

- the Lie algebra $\mathfrak{g}$ of isomorphism classes of objects,
- the vector space $\mathfrak{a}$ of endomorphisms of any object,
- a representation of $\mathfrak{g}$ on $\mathfrak{a}$,
- an element of $H^3(\mathfrak{g}, \mathfrak{a})$.

Suppose G is a simply-connected compact simple Lie group. Let $\mathfrak{g}$ be its Lie algebra. A lemma of Whitehead says:

$$H^3(\mathfrak{g}, \mathbb{R}) = \mathbb{R}$$

So:

Corollary. For any $k \in \mathbb{R}$ there is a Lie 2-algebra $\mathfrak{string}_k(\mathfrak{g})$ for which:

- $\mathfrak{g}$ is the Lie algebra of isomorphism classes of objects;
- $\mathbb{R}$ is the vector space of endomorphisms of any object.

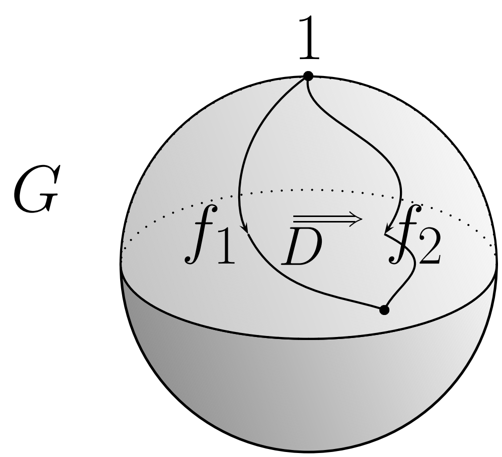
Every Lie 2-algebra with these properties is equivalent to $\mathfrak{string}_k(\mathfrak{g})$ for some unique $k \in \mathbb{R}$.

Theorem. For any $k \in \mathbb{Z}$, $\mathfrak{string}_k(\mathfrak{g})$ is the Lie 2-algebra of an infinite-dimensional Lie 2-group $\mathrm{String}_k(G)$.

An object of $\mathrm{String}_k(G)$ is a smooth path

$$f\colon [0, 2\pi] \to G$$

starting at the identity. A morphism from f_1 to f_2 is an equivalence class of pairs (D, α) where D is a disk going from f_1 to f_2 and $\alpha \in \mathrm{U}(1)$:



Any two such pairs (D_1, α_1) and (D_2, α_2) have a 3-ball B whose boundary is $D_1 \cup D_2$. The pairs are equivalent when

$$\exp\left(2\pi i k \int_B \nu\right) = \alpha_2/\alpha_1$$

where ν is the left-invariant closed 3-form on G with

$$\nu(x, y, z) = \langle [x, y], z \rangle$$

and $\langle \cdot, \cdot \rangle$ is the smallest invariant inner product on $\mathfrak{g}$ such that ν gives an integral cohomology class.

Theorem. The morphisms in $\mathrm{String}_k(G)$ starting at the constant path form the level-k central extension of the loop group ΩG:

$$1 \longrightarrow \mathrm{U}(1) \longrightarrow \widehat{\Omega_k G} \longrightarrow \Omega G \longrightarrow 1$$

For any category $\mathcal{C}$ there is a space $|\mathcal{C}|$, the **geometric realization of the nerve** of $\mathcal{C}$, built from a vertex for each object:

$$\bullet\ x$$

an edge for each morphism:

$$\bullet \xrightarrow{f} \bullet$$

a triangle for each composable pair of morphisms:

$$\begin{tikzcd} & \bullet \arrow[dr, "g"] & \\ \bullet \arrow[ur, "f"] \arrow[rr, "fg"'] & & \bullet \end{tikzcd}$$

a tetrahedron for each composable triple:

$$\begin{tikzcd} & \bullet \arrow[ddr, "g"] \arrow[drr, "gh"] & & \\ \bullet \arrow[ur, "f"] \arrow[rrr, dotted, "fgh"] \arrow[dr, "fg"'] & & & \bullet \\ & & \bullet \arrow[ur, "h"'] & \end{tikzcd}$$

and so on...

A 2-group is a category *with a product and inverses.* So, if $\mathcal{G}$ is a 2-group, $|\mathcal{G}|$ is a topological group.

More generally, we can define a topological group $|\mathcal{G}|$ for any *topological* 2-group $\mathcal{G}$.

Theorem. For any $k \in \mathbb{Z}$, there is a short exact sequence of topological groups

$$1 \longrightarrow K(\mathbb{Z}, 2) \longrightarrow |\mathrm{String}_k(G)| \xrightarrow{p} G \longrightarrow 1$$

where p is a fibration. Using this we can show:

$$\begin{aligned} \pi_1(|\mathrm{String}_k(G)|) &= 0 \\ \pi_2(|\mathrm{String}_k(G)|) &= \mathbb{Z}/k\mathbb{Z} \\ \pi_3(|\mathrm{String}_k(G)|) &= 0 \qquad\qquad \text{if } k \neq 0 \end{aligned}$$

Theorem. When $k = 1$, $|\mathrm{String}_k(G)|$ is the '3-connected cover' of G: the topological group formed by making the 3rd homotopy group of G trivial.

For example, start with $\mathrm{O}(n)$:

- Making π_0 trivial gives $\mathrm{SO}(n)$.
- Making π_1 trivial gives $\mathrm{Spin}(n)$.
- π_2 of $\mathrm{Spin}(n)$ is already trivial.
- Making π_3 trivial gives $\mathrm{String}(n)$.

We are claiming

$$\mathrm{String}(n) \simeq |\mathrm{String}_k(G)|$$

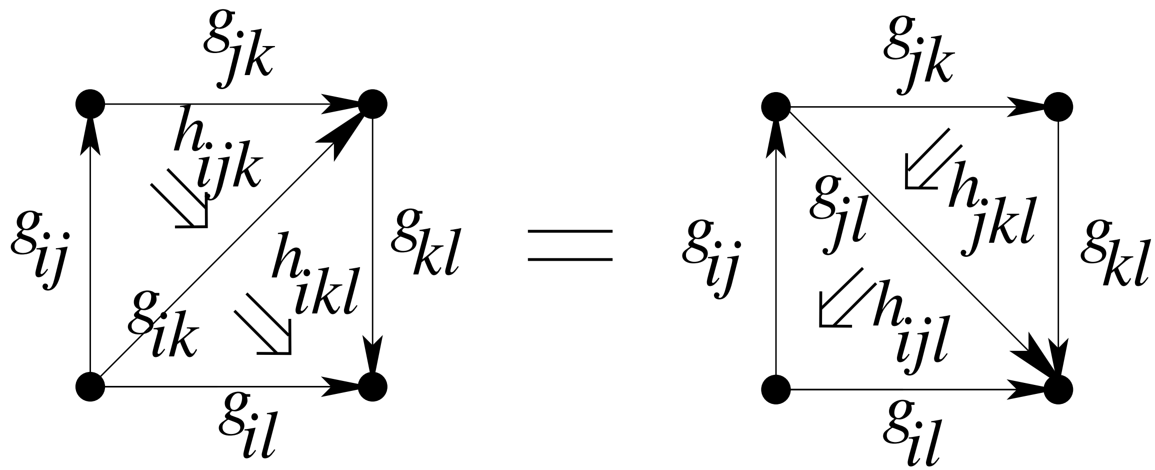
where $G = \mathrm{Spin}(n)$ and $k = 1$.

2-Bundles — Quick and Dirty

For any topological 2-group $\mathcal{G}$ and any space X, we can define a **principal $\mathcal{G}$-2-bundle over** X to consist of:

- an open cover U_i of X,
- continuous maps
$$g_{ij} \colon U_i \cap U_j \to \mathrm{Ob}(\mathcal{G})$$
satisfying $g_{ii} = 1$, and
- continuous maps
$$h_{ijk} \colon U_i \cap U_j \cap U_k \to \mathrm{Mor}(\mathcal{G})$$
with
$$h_{ijk}(x) \colon g_{ij}(x) g_{jk}(x) \to g_{ik}(x)$$

satisfying the **nonabelian 2-cocycle condition:**



on any quadruple intersection $U_i \cap U_j \cap U_k \cap U_\ell$.

There's a natural notion of 'equivalence' for 2-bundles over X, since they form a 2-category.

Theorem. For any topological 2-group $\mathcal{G}$ and paracompact Hausdorff space X, there is a 1-1 correspondence between:

- equivalence classes of principal $\mathcal{G}$-2-bundles over X,
- isomorphism classes of principal $|\mathcal{G}|$-bundles over X,
- homotopy classes of maps $f\colon X \to B|\mathcal{G}|$.

So, $B|\mathcal{G}|$ is the classifying space for $\mathcal{G}$-2-bundles.

We have homomorphisms

$$\mathrm{String}(n) \longrightarrow \mathrm{Spin}(n) \longrightarrow \mathrm{SO}(n) \longrightarrow \mathrm{O}(n)$$

Given an n-dimensional Riemannian manifold X, we can reduce the structure group of the frame bundle from $\mathrm{O}(n)$ to:

- $\mathrm{SO}(n)$ if we have an orientation on X,
- $\mathrm{Spin}(n)$ if we have a spin structure on X,
- $\mathrm{String}(n)$ if we have a string structure on X.

Corollary. For any Riemannian n-manifold X, a string structure on X gives a $\mathcal{G}$-2-bundle over X, where $\mathcal{G} = \mathrm{String}_k(G)$ with $G = \mathrm{Spin}(n)$ and $k = 1$.

2-Connections — Quick and Dirty

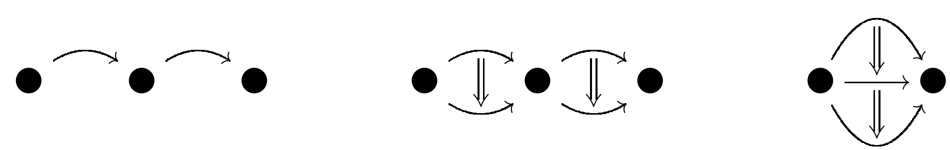
Let $\mathcal{G}$ be a Lie 2-group, P the trivial principal $\mathcal{G}$-2-bundle over some smooth manifold X. A **2-connection** on P assigns holonomies to paths in X:

$$\mathrm{hol}\colon\ x \xrightarrow{\gamma} y \quad \mapsto \quad \bullet \xrightarrow{\mathrm{hol}(\gamma)} \bullet \ \in \mathrm{Ob}(\mathcal{G})$$

and surfaces going between paths:

$$\mathrm{hol}\colon\ \begin{tikzcd} x \arrow[r, bend left=50, "\gamma"{name=U}] \arrow[r, bend right=50, "\eta"'{name=D}] & y \arrow[Rightarrow, from=U, to=D, "\Sigma"] \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} \bullet \arrow[r, bend left=50, "\mathrm{hol}(\gamma)"{name=U}] \arrow[r, bend right=50, "\mathrm{hol}(\eta)"'{name=D}] & \bullet \arrow[Rightarrow, from=U, to=D, "\mathrm{hol}(\Sigma)"] \end{tikzcd} \ \in \mathrm{Mor}(\mathcal{G})$$

in a manner preserving all 3 forms of composition:



Theorem. Let

$$\mathfrak{g}_0 \xleftarrow{\partial} \mathfrak{g}_1$$

be the infinitesimal crossed module obtained by differentiating the crossed module

$$G_0 \xleftarrow{\partial} G_1$$

corresponding to $\mathcal{G}$. Then there is a 1-1 correspondence between 2-connections on $P \to X$ and **connections**:

- a $\mathfrak{g}_0$-valued 1-form A on X
- a $\mathfrak{g}_1$-valued 2-form B on X

satisfying the **fake flatness** condition:

$$dA + \frac{1}{2}[A, A] + \partial B = 0$$

All this generalizes to nontrivial 2-bundles.

Nice Problem. When $\mathcal{G} = \mathrm{String}_k(G)$, compute the real characteristic classes of a $\mathcal{G}$-2-bundle in terms of an arbitrary connection on this 2-bundle.

The homomorphism $|\mathcal{G}| \xrightarrow{p} G$ gives an algebra homomorphism:

$$H^*(BG, \mathbb{R}) \xrightarrow{p^*} H^*(B|\mathcal{G}|, \mathbb{R})$$

When $k = 1$ this is onto, with kernel generated by the 'second Chern class' $c_2 \in H^4(BG, \mathbb{R})$.

In this case, the real characteristic classes of $\mathcal{G}$-2-bundles are just like those of G-bundles, but with the second Chern class killed!