Categorification

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Abstract

Categorification is the process of finding category-theoretic analogs of set-theoretic concepts by replacing sets with categories, functions with functors, and equations between functions by natural isomorphisms between functors, which in turn should satisfy certain equations of their own, called 'coherence laws'. Iterating this process requires a theory of 'n-categories', algebraic structures having objects, morphisms between objects, 2-morphisms between morphisms and so on up to n-morphisms. After a brief introduction to n-categories and their relation to homotopy theory, we discuss algebraic structures that can be seen as iterated categorifications of the natural numbers and integers. These include tangle n-categories, cobordism n-categories, and the homotopy n-types of the loop spaces $\Omega^k S^k$. We conclude by describing a definition of weak n-categories based on the theory of operads.

1 Introduction

The tongue-twisting term 'categorification' was invented by Crane [19, 20]. It refers to the process of finding category-theoretic analogs of ideas phrased in the language of set theory, using the following analogy between set theory and category theory:

elements	objects
equations between elements	isomorphisms between objects
sets	categories
functions	functors
equations between functions	natural isomorphisms between functors

1. Analogy between set theory and category theory

Just as sets have elements, categories have objects. Just as there are functions between sets, there are functors between categories. Interestingly, the proper analog of an equation between elements is not an equation between objects, but an isomorphism. More generally, the analog of an equation between functions is a natural isomorphism between functors.

For example, the category FinSet, whose objects are finite sets and whose morphisms are functions, is a categorification of the set $\mathbb{N}$ of natural numbers. The disjoint union and Cartesian product of finite sets correspond to the sum and product in $\mathbb{N}$, respectively. Note that while addition and multiplication in $\mathbb{N}$ satisfy various equational laws such as commutativity, associativity and distributivity, disjoint union and Cartesian product satisfy such laws *only up to natural isomorphism*.

If one studies categorification one soon discovers an amazing fact: many deep-sounding results in mathematics are just categorifications of facts we learned in high school! There is a good reason for this. All along, we have been unwittingly 'decategorifying' mathematics by pretending that categories are just sets. We 'decategorify' a category by forgetting about the morphisms and pretending that isomorphic objects are equal. We are left with a mere set: the set of isomorphism classes of objects.

To understand this, the following parable may be useful. Long ago, when shepherds wanted to see if two herds of sheep were isomorphic, they would look for an explicit isomorphism. In other words, they would line up both herds and try to match each sheep in one herd with a sheep in the other. But one day, along came a shepherd who invented decategorification. She realized one could take each herd and 'count' it, setting up an isomorphism between it and some set of 'numbers', which were nonsense words like 'one, two, three, ...' specially designed for this purpose. By comparing the resulting numbers, she could show that two herds were isomorphic without explicitly establishing an isomorphism! In short, by decategorifying the category of finite sets, the set of natural numbers was invented.

According to this parable, decategorification started out as a stroke of mathematical genius. Only later did it become a matter of dumb habit, which we are now struggling to overcome by means of categorification. While the historical reality is far more complicated, categorification really has led to tremendous progress in mathematics during the 20th century. For example, Noether revolutionized algebraic topology by emphasizing the importance of homology groups. Previous work had focused on Betti numbers, which are just the dimensions of the rational homology groups. As with taking the cardinality of a set, taking the dimension of a vector space is a process of decategorification, since two vector spaces are isomorphic if and only if they have the same dimension. Noether noted that if we work with homology groups rather than Betti numbers, we can solve more problems, because we obtain invariants not only of spaces, but also of maps. In modern parlance, the nth rational homology is a *functor* defined on the *category* of topological spaces, while the nth Betti number is a mere *function* defined on the *set* of isomorphism classes of topological spaces. Of course, this way of stating Noether's insight is anachronistic, since it came before category theory. Indeed, it was in Eilenberg and Mac Lane's subsequent work on homology that category theory was born!

Decategorification is a straightforward process which typically destroys information about the situation at hand. Categorification, being an attempt to recover this lost information, is inevitably fraught with difficulties. One reason is that when categorifying, one does not merely replace equations by isomorphisms. One also demands that these isomorphisms satisfy some new equations of their own, called 'coherence laws'. Finding the right coherence laws for a given situation is perhaps the trickiest aspect of categorification.

For example, a monoid is a set with a product satisfying the associative law and a unit element satisfying the left and right unit laws. The categorified version of a monoid is a 'monoidal category'. This is a category C with a product $\otimes\colon C \times C \to C$ and and unit object $1 \in C$. If we naively impose associativity and the left and right unit laws as equational laws, we obtain the definition of a 'strict' monoidal category. However, the philosophy of categorification suggests instead that we impose them only up to natural isomorphism. Thus, as part of the structure of a 'weak' monoidal category, we specify a natural isomorphism

$$a_{x,y,z} \colon (x \otimes y) \otimes z \to x \otimes (y \otimes z)$$

called the 'associator', together with natural isomorphisms

$$l_x \colon 1 \otimes x \to x,$$

$$r_x \colon x \otimes 1 \to x.$$

Using the associator one can construct isomorphisms between any two parenthesized versions of the tensor product of several objects. However, we really want a *unique* isomorphism. For example, there are 5 ways to parenthesize the tensor product of 4 objects, which are related by the associator as follows:

$$\begin{tikzcd}
((x \otimes y) \otimes z) \otimes w \arrow[r, "a_{x\otimes y,z,w}"] \arrow[d, "a_{x,y,z}\otimes w"'] & (x \otimes y) \otimes (z \otimes w) \arrow[r, "a_{x,y,z\otimes w}"] & x \otimes (y \otimes (z \otimes w)) \\
(x \otimes (y \otimes z)) \otimes w \arrow[rr, "a_{x,y\otimes z,w}"] & & x \otimes ((y \otimes z) \otimes w) \arrow[u, "x\otimes a_{y,z,w}"']
\end{tikzcd}$$

In the definition of a weak monoidal category we impose a coherence law, called the 'pentagon identity', saying that this diagram commutes. Similarly, we impose a coherence law saying that the following diagram built using a, l and r commutes:

$$\begin{tikzcd}
(1 \otimes x) \otimes 1 \arrow[rr, "a_{1,x,1}"] \arrow[d, "l_x\otimes 1"'] & & 1 \otimes (x \otimes 1) \arrow[d, "1\otimes r_x"] \\
x \otimes 1 \arrow[r, "r_x"] & x & 1 \otimes x \arrow[l, "l_x"']
\end{tikzcd}$$

This definition raises an obvious question: how do we know we have found all the right coherence laws? Indeed, what does 'right' even *mean* in this context? Mac Lane's coherence theorem [45] gives one answer to this question: the above coherence laws imply that any two isomorphisms built using a, l and r and having the same source and target must be equal.

Further work along these lines allow us to make more precise the sense in which $\mathbb{N}$ is a decategorification of FinSet. For example, just as $\mathbb{N}$ forms a monoid under either addition or multiplication, FinSet becomes a monoidal category under either disjoint union or Cartesian product if we choose the isomorphisms a, l, and r sensibly. In fact, just as $\mathbb{N}$ is a 'rig', satisfying all the ring axioms except those involving additive inverses, FinSet is what one might call a 'rig category'. In other words, it satisfies the rig axioms up to natural isomorphisms satisfying the coherence laws discovered by Kelly [41] and Laplaza [44], who proved a coherence theorem in this context. Just as the decategorification of a monoidal category is a monoid, the decategorification of any rig category is a rig. In particular, decategorifying the rig category FinSet gives the rig $\mathbb{N}$. This idea is especially important in combinatorics, where the best proof of an identity involving natural numbers is often a 'bijective proof': one that actually establishes an isomorphism between finite sets [37, 54].

While coherence laws can sometimes be justified retrospectively by coherence theorems, certain puzzles point to the need for a deeper understanding of the *origin* of coherence laws. For example, suppose we want to categorify the notion of 'commutative monoid'. The strictest possible approach, where we take a strict monoidal category and impose an equational law of the form $x \otimes y = y \otimes x$, is almost completely uninteresting. It is much better to start with a weak monoidal category equipped with a natural isomorphism $B_{x,y}\colon x \otimes y \to y \otimes x$ called the 'braiding', and then impose coherence laws called 'hexagon identities' saying that the following two diagrams commute:

$$\begin{tikzcd}
x \otimes (y \otimes z) \arrow[r, "a^{-1}_{x,y,z}"] \arrow[d, "B_{x,y\otimes z}"'] & (x \otimes y) \otimes z \arrow[r, "B_{x,y}\otimes z"] & (y \otimes x) \otimes z \arrow[d, "a_{y,x,z}"] \\
(y \otimes z) \otimes x \arrow[r, "a_{y,z,x}"'] & y \otimes (z \otimes x) & y \otimes (x \otimes z) \arrow[l, "y\otimes B_{x,z}"]
\end{tikzcd}$$

$$
\begin{tikzcd}
(x \otimes y) \otimes z \arrow[r, "a_{x,y,z}"] \arrow[d, "B_{x\otimes y,z}"'] & x \otimes (y \otimes z) \arrow[r, "x\otimes B_{y,z}"] & x \otimes (z \otimes y) \arrow[d, "a^{-1}_{x,z,y}"] \\
z \otimes (x \otimes y) \arrow[r, "a^{-1}_{z,x,y}"'] & (z \otimes x) \otimes y & (x \otimes z) \otimes y \arrow[l, "B_{x,z}\otimes y"]
\end{tikzcd}
$$

This gives the definition of a weak 'braided monoidal category'. If we impose an additional coherence law of the form $B^{-1}_{x,y} = B_{y,x}$, we obtain the definition of a 'symmetric monoidal category'. Both of these concepts are very important; which one is 'right' depends on the context. However, neither implies that every pair of parallel morphisms built using the braiding are equal. A good theory of coherence laws must naturally account for these facts.

The deepest insights into such puzzles have traditionally come from topology. In homotopy theory it causes problems to work with spaces equipped with algebraic structures satisfying equational laws, because one cannot transport such structures along homotopy equivalences. It is better to impose laws *only up to homotopy*, with these homotopies satisfying certain coherence laws, but again only up to homotopy, with these higher homotopies satisfying their own higher coherence laws, and so on. Coherence laws thus arise naturally in infinite sequences. For example, Stasheff [59] discovered the pentagon identity and a sequence of higher coherence laws for associativity when studying the algebraic structure possessed by a space that is homotopy equivalent to a loop space. Similarly, the hexagon identities arise as part of a sequence of coherence laws for spaces homotopy equivalent to double loop spaces, while the extra coherence law for symmetric monoidal categories arises as part of a sequence for spaces homotopy equivalent to triple loop spaces. The higher coherence laws in these sequences turn out to be crucial when we try to *iterate* the process of categorification.

Starting in the late 1960's, Boardman, Vogt [13, 14] and others developed the study of these higher coherence laws into a full-fledged theory of 'homotopy-invariant algebraic structures'. However, we have yet to attain a general and systematic theory of categorification, particularly when it comes to iterated categorification, which requires a good theory of 'n-categories'. The main goal of this paper is to outline our current understanding of categorification and to point out some directions for further study.

The plan of the paper is as follows. In Section 2 we give a quick tour of n-category theory. This theory is just beginning to be developed, and there are various alternative approaches which have not yet been reconciled, but here we leave out most of the technical details and sketch what we expect from any reasonable approach. In Section 3 we discuss in more detail the lessons homotopy theory has for n-category theory. We hope this section can be followed even by those who are not already experts on homotopy theory. In Section 4 we describe some algebraic structures that amount to iterated categorifications of the natural numbers and the integers. A large amount of interesting mathematics emerges from the study of these structures. In Section 5 we summarize our own approach to n-categories.

In a previous paper [4] we sketched a program of using n-categories to clarify the relationships between topological quantum field theory and more traditional approaches to algebraic topology. The present paper covers some aspects of this program in more detail, taking advantage of work that has been done in the meantime. Various other aspects are treated in a series of papers entitled 'Higher-Dimensional Algebra' [2, 5, 6, 8].

2 n-Categories

One philosophical reason for categorification is that it refines our concept of 'sameness' by allowing us to distinguish between isomorphism and equality. In a set, two elements are either the same or different. In a category, two objects can be 'the same in a way' while still being different. In other words, they can be isomorphic but not equal. Even more importantly, two objects can be the same in more than one way, since there can be different isomorphisms between them. This gives rise to the notion of the 'symmetry group' of an object: its group of automorphisms.

Consider, for example, the fundamental groupoid $\Pi_1(X)$ of a topological space X: the category with points of X as objects and homotopy classes of paths with fixed endpoints as morphisms. This category captures all the homotopy-theoretic information about X in dimensions ≤ 1 — or more precisely, its homotopy 1-type [11]. The group of automorphisms of an object x in this category is just the fundamental group $\pi_1(X, x)$. If we decategorify the fundamental groupoid of X, we forget *how* points in X are connected by paths, remembering only *whether* they are, and we obtain the set of components of X. This captures only the homotopy 0-type of X.

This example shows how decategorification eliminates 'higher-dimensional information' about a situation. Categorification is an attempt to recover this information. This example also suggests that we can keep track of the homotopy 2-type of X if we categorify further and distinguish between paths that are equal and paths that are merely isomorphic (i.e., homotopic). For this we should work with a '2-category' having points of X as objects, paths as morphisms, and certain equivalence classes of homotopies between paths as 2-morphisms.

In a marvelous self-referential twist, the definition of '2-category' is simply the categorification of the definition of 'category'! Like a category, a 2-category has a class of objects, but now for any pair x, y of objects there is no longer a set $\hom(x, y)$; instead, there is a category $\hom(x, y)$. Objects of $\hom(x, y)$ are called morphisms of C, and morphisms between them are called 2-morphisms of C. Composition is no longer a function, but rather a functor:

$$\circ \colon \hom(x, y) \times \hom(y, z) \to \hom(x, z).$$

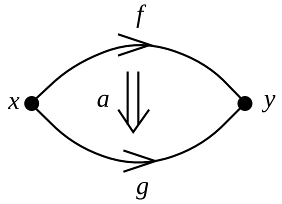
For any object x there is an identity $1_x \in \hom(x, x)$. And now we have a choice. On the one hand, we can impose associativity and the left and right unit laws strictly, as equational laws. If we do this, we obtain the definition of 'strict 2-category' [42]. On the other hand, we can impose them only up to natural isomorphism, with these natural isomorphisms satisfying the coherence laws discussed in the previous section. This is clearly more compatible with the spirit of categorification. If we do this, we obtain the definition of 'weak 2-category' [12]. (We warn the reader that strict 2-categories are traditionally known as '2-categories', while weak 2-categories are known as 'bicategories'. The present style of terminology, introduced by Kapranov and Voevodsky [40], has the advantage of generalizing easily to n-categories for arbitrary n.)

The classic example of a 2-category is Cat, which has categories as objects, functors as morphisms, and natural transformations as 2-morphisms. The presence of 2-morphisms gives Cat much of its distinctive flavor, which we would miss if we treated it as a mere category. Indeed, Mac Lane has said that categories were originally invented, not to study functors, but to study natural transformations! A good example of two functors that are not equal, but only naturally isomorphic, are the identity functor and the 'double dual' functor on the category of finite-dimensional vector spaces.

Given a topological space X, we can form a 2-category $\Pi_2(X)$ called the 'fundamental 2-groupoid' of X. The objects of this 2-category are the points of X. Given $x, y \in X$, the morphisms from x to y are the paths $f\colon [0,1] \to X$ starting at x and ending at y. Finally, given $f, g \in \hom(x, y)$, the 2-morphisms from f to g are the homotopy classes of paths in $\hom(x, y)$ starting at f and ending at g. Since the associative law for composition of paths holds only up to homotopy, this 2-category is a weak 2-category. One can prove that it captures the homotopy 2-type of X [18, 47]. If we decategorify the fundamental 2-groupoid of X, we obtain its fundamental groupoid.

From 2-categories it is a short step to dreaming of n-categories and even ω-categories — but it is not so easy to make these dreams into smoothly functioning mathematical tools. Roughly speaking, an n-category should be some sort of algebraic structure having objects, 1-morphisms between objects, 2-morphisms between 1-morphisms, and so on up to n-morphisms. There should be various ways of composing j-morphisms for $1 \leq j \leq n$, and these should satisfy various laws. As with 2-categories, we can try to impose these laws either strictly or weakly. Strict n-categories have been understood for quite some time now [23, 28], but more interesting for us are the weak ones. Various definitions of weak n-category are currently under active study [5, 10, 36, 57, 58, 61, 62, 63], and we discuss our own in Section 5. Here, however, we wish to sketch the main challenges any theory of weak n-categories must face, and some of the richness inherent in the notion of weak n-category.

Nota bene: Throughout the rest of this paper, 'n-category' will mean 'weak n-category' unless otherwise specified, and similarly for 'n-groupoid', etc.. For the sake of definiteness, we shall temporarily speak in terms of the 'globular' approach to n-categories developed by Batanin [10]. In this approach, for $j \geq 1$ any j-morphism $a\colon f \to g$ has a source f and a target g, which are $(j-1)$-morphisms. When $j \geq 2$ we require that f and g are parallel, by which we mean that $f, g\colon x \to y$ for some $(j-2)$-morphisms x and y. In this approach, we visualize a j-morphism as a j-dimensional disc. For example, for $j = 2$:



Other approaches to n-categories use j-morphisms with other shapes, such as simplices, discussed in Section 3, or opetopes, discussed in Section 5. We believe that there is basically a single notion of weak n-category lurking behind these different approaches. If this is true, they will eventually be shown to be equivalent, and choosing among them will be merely a matter of convenience. However, the precise meaning of 'equivalence' here is itself rather subtle and n-categorical in flavor [3, 5].

The first challenge to any theory of n-categories is to give an adequate treatment of coherence laws. Composition in an n-category should satisfy equational laws only at the top level, between n-morphisms. Any law concerning j-morphisms for $j < n$ should hold only 'up to equivalence'. Here a n-morphism is defined to be an 'equivalence' if it is invertible, while for $j < n$ a j-morphism is recursively defined to be an equivalence if it is invertible *up to equivalence*. Equivalence is generally the correct substitute for the notion of equality in n-categorical mathematics. When laws are formulated as equivalences, these equivalences should in turn satisfy coherence laws of their own, but again only up to equivalence, and so on. This becomes ever more complicated and unmanageable with increasing n unless one takes a systematic approach to coherence laws.

The second challenge to any theory of n-categories is to handle certain key examples. First, for any n, there should be an $(n+1)$-category $n\mathrm{Cat}$, whose objects are (small) n-categories, whose morphisms are suitably weakened functors between these, whose 2-morphisms are suitably weakened natural transformations, and so on. Here by 'suitably weakened' we refer to the fact that all laws should hold only up to equivalence. Second, for any topological space X, there should be an n-category $\Pi_n(X)$ whose objects are points of X, whose morphisms are paths, whose 2-morphisms are paths of paths, and so on, where we take homotopy classes only at the top level. $\Pi_n(X)$ should be an 'n-groupoid', meaning that all its j-morphisms are equivalences for $0 \leq j \leq n$. We call $\Pi_n(X)$ the 'fundamental n-groupoid of X'. Conversely, any n-groupoid should determine a topological space, its 'geometric realization'.

In fact, these constructions should render the study of n-groupoids equivalent to that of homotopy n-types (in a certain sense to be made precise in the next section).

A bit of the richness inherent in the concept of n-category becomes apparent when we make the following observation: an $(n + 1)$-category with only one object can be regarded as special sort of n-category. Suppose that C is an $(n+1)$-category with one object x. Then we can form the n-category $\tilde{C}$ by re-indexing: the objects of $\tilde{C}$ are the morphisms of C, the morphisms of $\tilde{C}$ are the 2-morphisms of C, and so on. The n-categories we obtain this way have extra structure. In particular, since the objects of $\tilde{C}$ are really morphisms in C from x to itself, we can 'multiply' (that is, compose) them.

The simplest example is this: if C is a category with a single object x, $\tilde{C}$ is the set of endomorphisms of x. This set is actually a monoid. Conversely, any monoid can be regarded as the monoid of endomorphisms of x for some category with one object x. We summarize this situation by saying that 'a one-object category is a monoid'. Similarly, a one-object 2-category is a monoidal category. It is natural to expect this pattern to continue in all higher dimensions; in fact, it is probably easiest to cheat and *define* a monoidal n-category to be an $(n + 1)$-category with one object.

Things get even more interesting when we iterate this process. Given an $(n + k)$-category C with only one object, one morphism, and so on up to one $(k - 1)$-morphism, we can form an n-category whose j-morphisms are the $(j + k)$-morphisms of C. In doing so we obtain a particular sort of n-category with extra structure and properties, which we call a 'k-tuply monoidal' n-category. Table 2 shows what we expect these to be like for low values of n and k. For example, the Eckmann-Hilton argument [4, 8, 27] shows that a 2-category with one object and one morphism is a commutative monoid. Categorifying this argument, one can show that a 3-category with one object and one morphism is a braided monoidal category. Similarly, we expect that a 4-category with one object, one morphism and one 2-morphism is a symmetric monoidal category, though this has not been worked out in full detail, because of our poor understanding of 4-categories. The fact that both braided and symmetric monoidal categories appear in this table seems to explain why both are natural concepts.

	$n=0$	$n=1$	$n=2$
$k=0$	sets	categories	2-categories
$k=1$	monoids	monoidal categories	monoidal 2-categories
$k=2$	commutative monoids	braided monoidal categories	braided monoidal 2-categories
$k=3$	‘ ’	symmetric monoidal categories	weakly involutory monoidal 2-categories
$k=4$	‘ ’	‘ ’	strongly involutory monoidal 2-categories
$k=5$	‘ ’	‘ ’	‘ ’

2. k-tuply monoidal n-categories

In any reasonable approach to n-categories there should be an $(n+1)$-category $n\mathrm{Cat}_k$ whose objects are k-tuply monoidal weak n-categories. One should also be able to treat $n\mathrm{Cat}_k$ as a full sub-$(n+k)$-category of $(n+k)\mathrm{Cat}$, though even for low n, k this is perhaps not as well known as it should be. Consider for example $n=0, k=1$. The objects of $0\mathrm{Cat}_1$ are one-object categories, or monoids. The morphisms of $0\mathrm{Cat}_1$ are functors between one-object categories, or monoid homomorphisms. But $0\mathrm{Cat}_1$ also has 2-morphisms corresponding to natural transformations. We leave it as an exercise to work out what these are in concrete terms. More recently, Kapranov and Voevodsky [40] have considered the case $n=k=1$.

We gave a detailed discussion of Table 2 in an earlier paper [4], and subsequent work by various authors has improved our understanding of some of the higher entries [8, 24, 26]. There are many interesting processes going from each entry in this table to its neighbors. We list some of the main ones below. Most of these have only been thoroughly studied for low values of n and k, often in the framework of ‘semistrict’ n-categories, which are a kind of halfway house between strict and weak ones. We expect that they all generalize to weak k-tuply monoidal n-categories for arbitrary n and k, but in many cases this has not yet been proved.

• *Decategorification:* $(n, k) \to (n-1, k)$. Let C be a k-tuply monoidal n-category C. Then there should be a k-tuply monoidal $(n-1)$-category $\mathrm{Decat}C$ whose j-morphisms are the same as those of C for $j < n-1$, but whose $(n-1)$-morphisms are isomorphism classes of $(n-1)$-morphisms of C.

• *Discrete categorification:* $(n, k) \to (n+1, k)$. There should be a 'discrete' k-tuply monoidal $(n+1)$-category $\mathrm{Disc}C$ having the j-morphisms of C as its j-morphisms for $j \le n$, and only identity $(n+1)$-morphisms. The decategorification of $\mathrm{Disc}C$ should be C.

• *Delooping:* $(n, k) \to (n+1, k-1)$. There should be a $(k-1)$-tuply monoidal $(n+1)$-category BC with one object obtained by reindexing, the j-morphisms of BC being the $(j+1)$-morphisms of C. We use the notation 'B' and call BC the 'delooping' of C because of its relation to the classifying space construction in topology.

• *Looping:* $(n, k) \to (n-1, k+1)$. Given objects x, y in an n-category, there should be an $(n-1)$-category $\hom(x, y)$. If $x = y$ this should be a monoidal $(n-1)$-category, and we denote it as $\mathrm{end}(x)$. For $k > 0$, if 1 denotes the unit object of the k-tuply monoidal n-category C, $\mathrm{end}(1)$ should be a $(k+1)$-tuply monoidal $(n-1)$-category. We call this process 'looping', and denote the result as ΩC, because of its relation to loop space construction in topology. For $k > 0$, looping should extend to an $(n+k)$-functor $\Omega\colon n\mathrm{Cat}_k \to (n-1)\mathrm{Cat}_{k+1}$. The case $k = 0$ is a bit different: we should be able to loop a 'pointed' n-category, one having a distinguished object x, by letting $\Omega C = \mathrm{end}(x)$. In either case, the j-morphisms of ΩC correspond to certain $(j-1)$-morphisms of C.

• *Forgetting monoidal structure:* $(n, k) \to (n, k-1)$. By forgetting the kth level of monoidal structure, we should be able to think of C as a $(k-1)$-tuply monoidal n-category FC. This should extend to an n-functor $F\colon n\mathrm{Cat}_k \to n\mathrm{Cat}_{k-1}$.

• *Stabilization:* $(n, k) \to (n, k+1)$. Though adjoint n-functors are still poorly understood, there should be a left adjoint to forgetting monoidal structure, which we call 'stabilization' and denote by $S\colon n\mathrm{Cat}_k \to n\mathrm{Cat}_{k+1}$.

(In our previous work we called it 'suspension', but this is probably a bit misleading.) The 'stabilization hypothesis' [4] states that for $k \geq n+2$, stabilization is an equivalence from $n\mathrm{Cat}_k$ to $n\mathrm{Cat}_{k+1}$. This is why the nth column of Table 2 has only $n+2$ distinct entries, and then settles down. While not yet proven or even formulated as a precise conjecture except in low dimensions, there is a lot of good evidence for this hypothesis, some of which we mention in the next section. In what follows, we assume this hypothesis and call a k-tuply monoidal n-category with $k = n+2$ a 'stable n-category'.

- *Forming the generalized center:* $(n,k) \to (n,k+1)$. Thinking of C as an object of the $(n+k)$-category $n\mathrm{Cat}_k$, there should be a $(k+1)$-tuply monoidal n-category ZC, the 'generalized center' of C, given by $\Omega^k(\mathrm{end}(C))$. In other words, ZC is the largest sub-$(n+k+1)$-category of $(n+k)\mathrm{Cat}$ having C as its only object, 1_C as its only morphism, 1_{1_C} as its only 2-morphism, and so on up to dimension k. This construction gets its name from the case $n = 0$, $k = 1$, where ZC is the usual center of the monoid C. Categorifying leads to the case $n = 1$, $k = 1$, which gives a very important construction of braided monoidal categories from monoidal categories [38, 40, 48]. In particular, when C is the monoidal category of representations of a Hopf algebra H, ZC is the braided monoidal category of representations of the quantum double $D(H)$. Categorifying still further, Baez and Neuchl [8] treated the case $n = 2$, $k = 1$. Subsequently Crans [24] corrected some errors in their work and dealt with the cases $n = 2$, $k > 1$.

3 Lessons from Homotopy Theory

In Grothendieck's famous 600-page letter to Quillen [35], he proposed developing n-category theory as a vast generalization of homotopy theory, with a special class of n-categories — the n-groupoids — corresponding to homotopy n-types. When this idea is finally worked out, we will be able to translate all of homotopy theory into the language of n-groupoids. Eventually this should deepen our understanding of the conceptual foundations of homotopy theory, and help us apply its techniques to other branches of mathematics. But even now, with n-category theory still in its squalling infancy, this translation project is worthwhile. The reason is that homotopy theory is our best source of insight into n-categories. We need to be careful here, since homotopy theory avoids precisely what is most new and interesting about general n-categories, namely the presence of j-morphisms that are not equivalences. However, this is a bit less of a drawback than it might at first seem. After all, the most mysterious aspect of n-category theory is the origin of coherence laws, and these, being implemented as equivalences, appear already in the context of n-groupoids.

ω-groupoids	homotopy types
n-groupoids	homotopy n-types
k-tuply groupal ω-groupoids	homotopy types of k-fold loop spaces
k-tuply groupal n-groupoids	homotopy n-types of k-fold loop spaces
k-tuply monoidal ω-groupoids	homotopy types of E_k spaces
k-tuply monoidal n-groupoids	homotopy n-types of E_k spaces
stable ω-groupoids	homotopy types of infinite loop spaces
stable n-groupoids	homotopy n-types of infinite loop spaces
$\mathbb{Z}$-groupoids	homotopy types of spectra

3. Translating between n-groupoid theory and homotopy theory

We can begin this translation project with the help of Table 3. In explaining this table, we shall use the simplicial approach to n-groupoids instead of the globular approach. The reason is that simplicial methods are quite popular among topologists, so all the necessary machinery has already been developed [49]. For other approaches to n-groupoids, see the work of Brown and his collaborators [16], Tamsamani [63], and Batanin [10].

In topology one usually speaks of 'j-cells' rather than j-morphisms. A 'simplicial set' has a set of j-cells for each $j \geq 0$, which we visualize as being shaped like j-simplices. For each j-cell f there are certain $(j-1)$-cells $d_0 f, \ldots, d_n f$ called 'faces' and $(j+1)$-cells $i_0 f, \ldots, i_{n+1} f$ called 'degeneracies'.

One also requires that the face and degeneracy maps satisfy certain well-known relations; we will not need the formulas here.

A 'j-dimensional horn' in a simplicial set is, roughly speaking, a configuration in which all but one of the faces of a j-simplex have been filled in by $(j-1)$-cells in a consistent way. A simplicial set for which any horn can be extended to a j-cell is called a 'Kan complex'. A Kan complex is the simplicial version of an ω-groupoid: a structure like an n-groupoid, but without any cutoff on the dimension of the j-morphisms.

To see how this idea works, suppose we have a Kan complex containing a 'composable' pair of 1-cells f and g, meaning that $d_1 f = d_0 g$. This gives a 2-dimensional horn with f and g as two of its faces, so we can extend this horn to a 2-cell F, which has as its third face some 1-cell h:

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, dashed, "h"'] & \arrow[u, phantom, "F\Downarrow"] & z \end{tikzcd}$$

In this situation, we call h 'a composite' of f and g, and think of F as the 'process of composition'. Note that there is not a unique preferred composite, so composition is not an operation in the traditional sense. However, any two composites can be seen to be equivalent, where two j-cells with all the same faces are said to be 'equivalent' if there is a $(j+1)$-cell having them as two of its faces, the rest being degenerate. From an algebraic viewpoint, the reason is that we have defined composition by a *universal property*.

Thanks to the magic of universal properties, Kan complexes are a wonderfully efficient formalism for studying ω-groupoids. In particular, there is no need to explicitly list coherence laws! They are all implicit in the fact that every horn can be extended to a cell, and they automatically become explicit if we arbitrarily choose processes of composition. For example, given a composable triple of 1-cells, one obtains the associator by considering a 3-simplex with these as three of its edges and making clever use of the horn-filling condition. Likewise, given a composable quadruple of 1-cells, one obtains the pentagon coherence law by considering a suitable 4-simplex. In fact, all the higher coherence laws for associativity, which Stasheff [59] organized into polyhedra called 'associahedra', have been obtained from higher-dimensional simplices by Street [61] in his simplicial approach to ω-categories.

If we take the liberty of calling Kan complexes 'ω-groupoids', we can set up a correspondence between ω-groupoids and homotopy types as follows. Given a topological space X, we can form an ω-groupoid $\Pi(X)$ whose j-cells are all the continuous maps from the standard j-simplex into X, with faces and degeneracies defined in the obvious way. We think of this as the 'fundamental ω-groupoid of X'. Conversely, given an ω-groupoid G, we can form a topological space by taking one geometrical j-simplex for each j-cell of G and gluing them all together using the face and degeneracy maps in the obvious way. This is called the 'geometric realization of G' and denoted $|G|$.

We thus obtain functors going both ways between the category $\omega\mathrm{Gpd}$, having Kan complexes as objects and simplicial maps between these as morphisms, and the category Top, having nice spaces as objects and continuous maps as morphisms. (We say a space is 'nice' if it is compactly generated and homotopy equivalent to a CW complex; we use this nonstandard definition of Top to exclude various pathologies.) While these functors are adjoint to one another, they do not set up an equivalence of categories. Nonetheless, we expect that $\omega\mathrm{Gpd}$ and Top are 'the same' in a subtler sense — namely, as ω-categories. More precisely, these categories should extend to ω-categories, where the 2-morphisms correspond to homotopies between maps, the 3-morphisms correspond to homotopies between homotopies, and so on. The functors

$$\Pi\colon \mathrm{Top} \to \omega\mathrm{Gpd},$$

$$|\cdot|\colon \omega\mathrm{Gpd} \to \mathrm{Top}$$

should then extend to ω-functors, giving an equivalence of ω-categories.

In the absence of a good theory of ω-categories, topologists have traditionally used other language to express the fact that $\omega\mathrm{Gpd}$ and Top are 'the same' for the purposes of homotopy theory. For example, Π and $|\cdot|$ establish an equivalence between the homotopy category of ω-groupoids and the homotopy category of nice spaces. Here the 'homotopy category' is formed by adjoining formal inverses to all maps inducing isomorphisms of homotopy groups [32], where one defines the homotopy groups of a Kan complex to be those of its geometric realization. An object in the homotopy category of Top is called a 'homotopy type' [11].

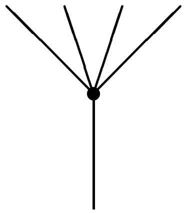
Starting from the correspondence between ω-groupoids and homotopy types we can set up the other correspondences listed in Table 3. For example, we can define an 'n-groupoid' to be a Kan complex such that for $j > n+1$ any configuration in which all the faces of a j-simplex have been filled in by $(j-1)$-cells in a consistent way can be uniquely extended to a j-cell. This ensures that all cells of dimension higher than n play the role of equations. The geometric realization of an n-groupoid is a space with vanishing homotopy groups above dimension n, and the homotopy category of such spaces is called the category of 'homotopy n-types'. The homotopy category of n-groupoids is equivalent to the category of homotopy n-types, and in fact one expects an equivalence of $(n+1)$-categories.

We define a 'k-tuply groupal ω-groupoid' to be a Kan complex with only one j-cell for $j < k$. Under the correspondence between ω-groupoids and homotopy types, these correspond to homotopy types with vanishing homotopy groups below dimension k. Similarly, we define a 'k-tuply groupal n-groupoid' to be an $(n+k)$-groupoid with only one j-cell for $j < k$. These correspond to homotopy $(n+k)$-types with vanishing homotopy groups below dimension k.

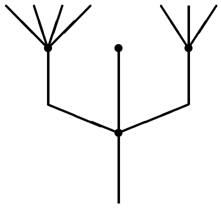
Recall from the previous section that we can think of an n-category C with only one j-morphism for $j < k$ as an n-category with extra structure and properties. In the globular approach we do this by 'reindexing', constructing an n-category whose j-morphisms are the $(j+k)$-morphisms of C. Similarly, it is often useful to think of a k-tuply groupal n-groupoid G as an n-groupoid with extra structure and properties. However, in the simplicial approach we cannot simply reindex the cells of G. Instead, we can use a standard construction [49] to form a Kan complex $\Omega^k G$ whose geometric realization is homotopy equivalent to the kth loop space of the geometric realization of G. This is why in Table 3 we say that k-tuply groupal n-groupoids correspond to homotopy n-types of k-fold loop spaces. Similarly, k-tuply groupal ω-groupoids correspond to homotopy types of k-fold loop spaces.

Exactly what extra structure and properties does an ω-groupoid have if it is k-tuply groupal? In other words, what extra structure and properties does a space have if it is homotopy equivalent to a k-fold loop space? This question has inspired the development of many interesting mathematical tools. Here we shall describe just one of these, the 'little k-cubes operad', invented by Boardman and Vogt [13, 14] and cast into the language of operads by May [50].

We begin with the definition of an 'operad'. For each $\ell \geq 0$, an operad O has a set O_ℓ of 'ℓ-ary operations'. We visualize such an operation as a tree with one vertex or 'node', ℓ edges representing inputs coming in from above, and one edge representing the output coming out from below:



We can compose these trees by feeding the outputs of ℓ of them into one with ℓ inputs:



In other words, for any $i_1, \ldots, i_\ell$ there is a function

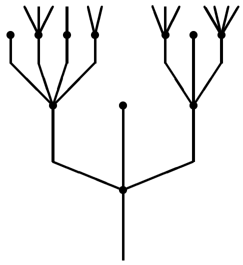
$$\begin{array}{rcl} O_\ell \times O_{i_1} \times \cdots \times O_{i_\ell} & \to & O_{i_1 + \cdots + i_\ell}. \\ (f, g_1, \ldots, g_\ell) & \mapsto & f \cdot (g_1, \ldots, g_\ell) \end{array}$$

We require that composition is 'associative', meaning that

$$f \cdot (g_1 \cdot (h_{11}, \ldots, h_{1i_1}), \ldots, g_\ell \cdot (h_{\ell 1}, \ldots, h_{\ell i_\ell})) =$$

$$(f \cdot (g_1, \ldots g_\ell)) \ \cdot \ (h_{11}, \ldots, h_{1i_1}, \ldots\ldots, h_{\ell 1}, \ldots, h_{\ell i_\ell})$$

whenever both sides are well-defined. This makes composites such as the following one unambiguous:



We also require the existence of an unit operation $1 \in O_1$ such that

$$1 \cdot (f) = f, \qquad f \cdot (1, \ldots, 1) = f$$

for all $f \in O_\ell$.

What we have so far is an *planar operad*. For a full-fledged operad, we also assume that there are right actions of the symmetric groups S_ℓ on the sets O_ℓ for which the following compatibility conditions hold. First, for any $f \in O_\ell$, $\sigma \in S_\ell$, and $g_j \in O_{i_j}$ for $1 \le j \le \ell$, we require

$$(f\sigma) \cdot (g_{\sigma(1)}, \ldots, g_{\sigma(\ell)}) = (f \cdot (g_1, \ldots, g_\ell))\, \rho(\sigma),$$

where

$$\rho\colon S_\ell \to S_{i_1 + \cdots + i_\ell}$$

is the obvious homomorphism. Second, for any $f \in O_\ell$, and $g_j \in O_{i_j}$, $\sigma_j \in S_{i_j}$ for $1 \le j \le \ell$, we require

$$f \cdot (g_1\sigma_1, \ldots, g_\ell\sigma_\ell) = (f \cdot (g_1, \ldots, g_\ell))\, \rho'(\sigma_1, \ldots, \sigma_\ell),$$

where

$$\rho'\colon S_{i_1} \times \cdots \times S_{i_\ell} \to S_{i_1 + \cdots + i_\ell}$$

is the obvious homomorphism.

Just as groups are interesting for their actions, operads are interesting for their 'algebras'. Given an operad O, an 'O-algebra' is a set A equipped with actions

$$\alpha\colon O_\ell \times A^\ell \to A,$$

or equivalently, maps

$$\alpha\colon O_\ell \to \hom(A^\ell, A)$$

representing the ℓ-ary operations of O as actual operations on A. We require that α sends the identity operation $1 \in O_1$ to the identity function on A and sends composites to composites:

$$\alpha(f \cdot (g_1, \ldots, g_\ell)) = \alpha(f) \circ (\alpha(g_1) \times \cdots \times \alpha(g_\ell)).$$

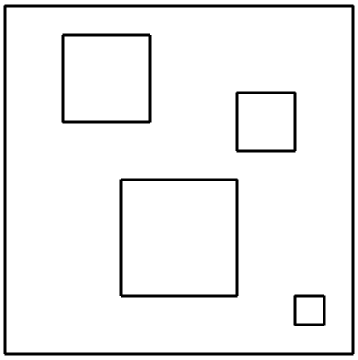
We also require that

$$\alpha(f\sigma) = \alpha(f)\sigma,$$

where $f \in O_\ell$, and $\sigma \in S_\ell$ acts on $\hom(A^\ell, A)$ on the right by permuting the factors in A^ℓ. We omit this requirement if O is merely planar.

More generally, one can define operads and their algebras in any symmetric monoidal category C, which amounts to replacing sets and functions in the above definitions by objects and morphisms in C, and replacing the Cartesian product by the tensor product in C. We shall mainly be interested in operads in the category Top, which are called 'topological operads'. Spaces with extra structure and properties can often be described as algebras of topological operads.

The most interesting example for us is the 'little k-cubes operad', $C(k)$. Here the space $C(k)_\ell$ of ℓ-ary operations is the space of ℓ disjoint k-cubes linearly embedded in the cube $[0,1]^k$ with their edges parallel to the coordinate axes:



An algebra of the little k-cubes operad is called an 'E_k space'. Given a space X with a basepoint, there is an obvious way to make $\Omega^k X$ into an E_k space using the fact that it consists of maps $f\colon [0,1]^k \to X$ sending the boundary of the cube to the basepoint. Conversely, the set of components of an E_k space automatically becomes a monoid, and if this monoid is a group then the E_k space is homotopy equivalent to a k-fold loop space. Homotopy types of E_k spaces are thus a bit more general than homotopy types of k-fold loop spaces. While the latter correspond to k-tuply groupal ω-groupoids, the former should correspond to 'k-tuply monoidal ω-groupoids', that is, ω-categories with only one j-morphism for $j < k$, for which all j-morphisms are equivalences for $j > k$ — but not necessarily for $j = k$. There should similarly be a correspondence between k-tuply monoidal n-groupoids and homotopy n-types of E_k spaces.

At the end of the previous section we listed various processes going between neighboring entries of Table 2. When restricted to k-tuply groupal n-groupoids, most of these processes have well-known topological interpretations, which we summarize below. We let $n\mathrm{Type}_k$ stand for the category of homotopy n-types of k-fold loop spaces. It is easiest to define this as the full subcategory of the homotopy category of nice spaces X with basepoint such that $\pi_j(X) = 0$ unless $k \le j \le n+k$. However, by repeated looping, we shall think of the objects of $n\mathrm{Type}_k$ as k-fold loop spaces with vanishing homotopy groups above dimension n. A technical point worth noting is that for $k = 0$, these really correspond to n-groupoids *equipped with a distinguished 0-cell.*

- *Decategorification:* $(n, k) \to (n-1, k)$. Let X be a k-fold loop space with $\pi_j(X) = 0$ for $j > n$. Then we can kill off its nth homotopy group by attaching cells, obtaining a k-fold loop space $\mathrm{Decat}X$ with $\pi_j(X) = 0$ for $j > n-1$. There is a map $X \to \mathrm{Decat}X$ inducing isomorphisms $\pi_j(X) \cong \pi_j(\mathrm{Decat}X)$ for $j \le n-1$. This process gives a functor $\mathrm{Decat}\colon n\mathrm{Type}_k \to (n-1)\mathrm{Type}_k$.

- *Discrete categorification:* $(n, k) \to (n+1, k)$. The forgetful functor $\mathrm{Disc}\colon n\mathrm{Type}_k \to (n+1)\mathrm{Type}_k$ is the left adjoint of Decat.

- *Delooping:* $(n, k) \to (n+1, k-1)$. There is a $(k-1)$-fold loop space BX, called the 'classifying space' or 'delooping' of X, with $\pi_j(BX) \cong \pi_{j-1}(X)$. This process gives a functor $B\colon n\mathrm{Type}_k \to (n+1)\mathrm{Type}_{k-1}$.

- *Looping:* $(n, k) \to (n-1, k+1)$. For $k > 0$, there is a $(k+1)$-fold loop space ΩX, consisting of all based loops in X, with $\pi_j(\Omega X) \cong \pi_{j-1}(X)$. This process gives a functor $\Omega\colon n\mathrm{Type}_k \to (n-1)\mathrm{Type}_{k+1}$ that is right adjoint to B. For $k > 0$, delooping followed by looping is naturally isomorphic to the identity functor.

• *Forgetting monoidal structure:* $(n, k) \to (n, k-1)$. There is a forgetful functor $F\colon n\mathrm{Type}_k \to n\mathrm{Type}_{k-1}$.

• *Stabilization:* $(n, k) \to (n, k+1)$. There is a left adjoint to F called 'stabilization', which we denote by $S\colon n\mathrm{Type}_k \to n\mathrm{Type}_{k+1}$. If we define the 'suspension' functor $\Sigma\colon (n-1)\mathrm{Type}_k \to n\mathrm{Type}_k$ to be the left adjoint of the composite

$$n\mathrm{Type}_k \xrightarrow{\Omega} (n-1)\mathrm{Type}_{k+1} \xrightarrow{F} (n-1)\mathrm{Type}_k$$

then stabilization is naturally isomorphic to suspension followed by looping. The Freudenthal suspension theorem says that stabilization is an equivalence for $k \geq n+2$.

• *Forming the generalized center:* $(n, k) \to (n, k+1)$. This process needs to be adapted to stay within the world of k-tuply groupal groupoids, or in other words, homotopy n-types of k-fold loop spaces. We let $\mathrm{aut}(X)$ be the automorphism group of X as an object of $n\mathrm{Type}_k$, and define $ZX = \Omega^k(\mathrm{aut}(X))$.

The fact that stabilization is an equivalence for $k \geq n+2$ leads us to define a 'stable n-groupoid' to be an $(n+2)$-tuply groupal n-groupoid. We expect, in fact, that there is an $(n+1)$-category of stable n-groupoids, and that the decategorification of this is the n-category of stable $(n-1)$-groupoids. If we could take the inverse limit in a suitable sense, we would hope to obtain an ω-category of 'stable ω-groupoids'. These should correspond to what topologists call 'infinite loop spaces', an infinite loop space being a sequence of spaces $X_0, X_1, \ldots$, equipped with homeomorphisms $f_k\colon X_k \to \Omega X_{k+1}$.

Infinite loop spaces play an important role in stable homotopy theory [1, 29, 50]. A closely related concept is that of a 'spectrum'. Just as an infinite loop space should correspond to a stable ω-groupoid, a spectrum should correspond to a '$\mathbb{Z}$-groupoid', some sort of gadget with j-morphisms for all $j \in \mathbb{Z}$, all of which are equivalences. In fact, strict $\mathbb{Z}$-categories are easily defined in the globular approach, and strict $\mathbb{Z}$-groupoids then work out to be the same as $\mathbb{Z}$-graded chain complexes of abelian groups. Spectra can be viewed as a generalization of such chain complexes. Apart from what we know about spectra, however, the theory of 'weak $\mathbb{Z}$-categories' remains largely *terra incognita*.

4 Examples of Categorification

In what follows, we consider iterated categorifications of the natural numbers, and some variations on this theme where we adjoin formal inverses or duals. We shall see that quite a bit of mathematics amounts to the study of the resulting objects, some of which are astoundingly complicated. The more 'n-groupoidal' examples are familiar from homotopy theory, but the importance of some of the more 'n-categorical' ones has only become clear in recent work on topological quantum field theory.

The natural numbers are the free monoid on one element, while the integers are the free group on one element. Some categorified analogs of these notions are listed in Table 4. In the rest of this section, we work through these examples in detail.

We begin with the simplest algebraic structure of all: sets. The free set on one element is just the one-element set, denoted by 1. Note that here we are using the word 'the' in a generalized sense. Since all sets with the same cardinality are isomorphic, any singleton $\{x\}$ can be regarded as 'the' 1-element set. In a set, when we speak of 'the' element with some property, we imply that any other element having this property is *equal* to this one, but in a category, when we speak of 'the' object with some property, we mean that any other object having this property is *isomorphic* to this one — typically by means of a uniquely specified isomorphism. More generally, when we speak of 'the' object of an n-category having some property, we mean that any other object with this property is *equivalent* — typically by means of an equivalence which is specified uniquely up to an equivalence which is specified uniquely up to … and so on. This recursive weakening of the notion of uniqueness, and therefore of the meaning of 'the', is fundamental to categorification.

sets	1
monoids	$\mathbb{N}$
groups	$\mathbb{Z}$
k-tuply monoidal n-categories	$n\mathrm{Braid}_k$
k-tuply monoidal ω-categories	Braid_k
stable n-categories	$n\mathrm{Braid}$
stable ω-categories	Braid_∞
k-tuply monoidal n-categories with duals	$n\mathrm{Tang}_k$
stable n-categories with duals	$n\mathrm{Cob}$
k-tuply groupal n-groupoids	$\Pi_n(\Omega^k S^k)$
k-tuply groupal ω-groupoids	$\Pi(\Omega^k S^k)$
stable ω-groupoids	$\Pi(\Omega^\infty S^\infty)$
$\mathbb{Z}$-groupoids	the sphere spectrum

4. Algebraic structures and the free such structures on one generator

Next consider the free category on one object. This is just the category with one object x and one morphism $1_x\colon x \to x$. More generally, for each n, consider the free n-category on one object. In the globular approach to n-categories we may take this to be the n-category with one object x, one morphism $1_x\colon x \to x$, one 2-morphism $1_{1_x}\colon 1_x \Rightarrow 1_x$, and so on up to one n-morphism. This is an n-groupoid, namely the globular version of the fundamental n-groupoid of a point. By the remarks in the previous paragraph, the fundamental n-groupoid of any contractible space may also be considered 'the free n-category on one object'.

Things become more interesting when we generalize further and consider the free k-tuply monoidal n-category on one object, which we denote by $n\mathrm{Braid}_k$, for reasons soon to be apparent. Let us see what this looks like in the simplest cases, namely $n = 0$ and 1. The stabilization hypothesis says that we only need to consider $k \leq n + 2$.

- $0\mathrm{Braid}_0$, the free set on one element. This is the one-element set, 1.

• $0\mathrm{Braid}_1$, the free monoid on one element. This is the natural numbers, $\mathbb{N}$, with addition as its monoid structure.

• $0\mathrm{Braid}_2$, the free commutative monoid on one element. This is again $\mathbb{N}$, now regarded as a commutative monoid with addition as its monoid structure.

• $1\mathrm{Braid}_0$, the free category on one object. This is the category with one object and one morphism.

• $1\mathrm{Braid}_1$, the free monoidal category on one object x. The objects of this category are the tensor powers $x^{\otimes \ell}$, and the only morphisms are identity morphisms. (Here we are using the Mac Lane coherence theorem to make $1\mathrm{Braid}_1$ into a strict monoidal category.) This is the discrete categorification of $0\mathrm{Braid}_1$.

• $1\mathrm{Braid}_2$, the free braided monoidal category on one object x. This is the braid groupoid. The objects of this groupoid are the tensor powers $x^{\otimes \ell}$, and the only morphisms are automorphisms, with $\mathrm{end}(x^{\otimes \ell})$ being the ℓ-strand braid group B_ℓ, which has generators σ_i $(1 \leq i \leq \ell - 1)$ and relations

$$\begin{array}{rcll} \sigma_i \sigma_{i+1} \sigma_i & = & \sigma_{i+1} \sigma_i \sigma_{i+1}, & \\ \sigma_i \sigma_j & = & \sigma_j \sigma_i & \qquad \text{for } |i-j| > 1. \end{array}$$

• $1\mathrm{Braid}_3$, the free symmetric monoidal category on one object x. This is the symmetric groupoid. The objects of this groupoid are the tensor powers $x^{\otimes \ell}$, and the only morphisms are automorphisms, with $\mathrm{end}(x^{\otimes \ell})$ being the symmetric group on ℓ letters, S_ℓ. This group can be given a presentation like that of B_ℓ, but with the additional relations

$$\sigma_i^2 = 1.$$

The symmetric groupoid is equivalent to the category with finite sets as objects and bijections as morphisms. Note that, like the category FinSet having finite sets as objects and arbitrary functions as morphisms, this category is a rig category with $\mathbb{N}$ as its decategorification.

The following heuristic argument allows us to guess a general formula for $n\mathrm{Braid}_k$. In all the cases considered above, $n\mathrm{Braid}_k$ is a k-tuply monoidal

n-groupoid. This should hold in general, since for $j > 0$ the j-morphisms of $n\mathrm{Braid}_k$ arise from coherence laws, and should thus be equivalences. We also expect that $n\mathrm{Braid}_k$ is the 'free k-tuply monoidal n-groupoid on one object'. By Table 3, $n\mathrm{Braid}_k$ should thus be the fundamental n-groupoid of some E_k space $X_{n,k}$ with vanishing homotopy groups above dimension n. We also expect that $\mathrm{Decat}((n+1)\mathrm{Braid}_k) \simeq n\mathrm{Braid}_k$, so there should be some sort of inverse limit, the 'free k-tuply monoidal ω-groupoid on one object', which we denote by Braid_k. Corresponding to this there should be a sequence of maps $X_{n+1,k} \to X_{n,k}$, with $X_{n,k}$ obtained from $X_{n+1,k}$ by killing its $(n+1)$st homotopy group, and with the inverse limit being a space X_k whose fundamental ω-groupoid is Braid_k. By the correspondence between k-tuply monoidal ω-groupoids and E_k spaces, we expect that X_k is the 'free E_k space on one point'.

While this argument involves many forms of reasoning that have not yet been made rigorous, there is at least a precise meaning to the 'free E_k space on one point'. Given any operad O in the category of pointed spaces and any pointed space X, there is a standard construction of the 'free O-algebra on X', due to May [50]. To form the free O-algebra on one point, we should first form the free free pointed space on one point, namely S^0, and then apply this standard construction. The result is the disjoint union

$$\coprod_{\ell=0}^{\infty} O_\ell / S_\ell$$

which becomes an O-algebra in a tautologous way.

In the case at hand, since the the operad for E_k spaces is the little k-cubes operad $C(k)$, the free E_k space on one point is

$$X_k = \coprod_{\ell=0}^{\infty} C(k)_\ell / S_\ell.$$

We expect, therefore, an equivalence

$$n\mathrm{Braid}_k \simeq \Pi_n(X_k).$$

To work with this equivalence, it is helpful to note that $C(k)_\ell$ is homotopy equivalent to the 'configuration space' [55] of ℓ distinct points in the k-cube:

$$\{(x_1, \ldots, x_\ell) \in [0,1]^k : \, x_i \neq x_j \text{ if } i \neq j\}.$$

Moreover, this homotopy equivalence is compatible with the obvious actions of S_ℓ. It follows that X_k is homotopy equivalent to the pointed space of all finite sets of distinct points in the k-cube, where the empty set plays the role of basepoint.

To see how this works in an example, consider the case $n = 1$, $k = 2$. The space X_2 is equivalent to the space of finite sets of points in the square. A path in X_2 amounts to a braid with an arbitrary number of strands. It follows that the fundamental groupoid of X_2 is equivalent to the braid groupoid, so

$$1\mathrm{Braid}_2 \simeq \Pi_1(X_2).$$

More generally, in the globular approach to n-categories, the n-morphisms of $n\mathrm{Braid}_k$ should correspond to certain n-dimensional surfaces in $[0,1]^{n+k}$, which we could call 'n-braids in $n+k$ dimensions'. We see here an instance of a general theme, namely that in a k-tuply monoidal n-category the number n often plays the role of 'dimension', while k plays the role of 'codimension'.

Now let us turn to the 'free k-tuply groupal n-groupoid on one object', which we temporarily denote by $G_{n,k}$. For $n = 0$, $k = 1$ this is just the free group on one element, namely the integers, $\mathbb{Z}$. For higher values of n and k we may thus regard $G_{n,k}$ as a categorified, stabilized version of $\mathbb{Z}$. In what follows we restrict attention to the case $k > 0$, since in this case a k-tuply groupal n-groupoid automatically has a distinguished object, the unit.

Again we can use a heuristic argument to guess a formula for $G_{n,k}$. Since we expect that $G_{n,k} \simeq \mathrm{Decat} G_{n+1,k}$, there should be some sort of inverse limit G_k, the 'free k-tuply groupal ω-groupoid on one object'. By Table 3 we expect G_k to be the fundamental ω-groupoid of some k-fold loop space, so the problem is to determine this space. Just as the group $\mathbb{Z}$ is obtained from the monoid $\mathbb{N}$ by adjoining formal inverses, G_k should be obtained from Braid_k by adjoining formal weak inverses for all objects. More generally, we should be able to turn any k-tuply monoidal ω-groupoid into a k-tuply groupal ω-groupoid by adjoining formal weak inverses of objects. In the language of homotopy theory, this process should turn E_k spaces into k-fold loop spaces. In fact, this process is familiar in homotopy theory under the name of 'group completion' [9]. Since $n\mathrm{Braid}_k$ is the fundamental ω-groupoid of X_k, we thus expect G_k to be the fundamental ω-groupoid of the group completion of X_k. The group completion of X_k is homotopy equivalent to $\Omega^k S^k$, so we expect an equivalence

$$G_{n,k} \simeq \Pi_n(\Omega^k S^k).$$

Group completion automatically gives a map from X_k to $\Omega^k S^k$ which induces a k-tuply monoidal n-functor

$$\Pi_n(X_k) \to \Pi_n(\Omega^k S^k).$$

If the above guesses are correct, n-braids and the homotopy types of spheres play a fundamental role in n-category theory. Since the homotopy groups of spheres are notoriously hard to compute, this means that n-category theory has a certain built-in complexity. Perhaps we should amplify on this a bit. Suppose that C is a globular $(n+k)$-category and x is any object of C. Let $1_1 = 1_x$ and recursively define 1_{i+1} to be 1_{1_i}. Then we are claiming that any k-morphism $f\colon 1_{k-1} \to 1_{k-1}$ determines, at least up to equivalence, a k-tuply monoidal n-functor

$$n\mathrm{Braid}_k \to \mathrm{end}(1_{k-1})$$

mapping the generator of $n\mathrm{Braid}_k$ to f. Moreover, if f is an equivalence, we claim this factors through a k-tuply groupal n-functor

$$\Pi_n(\Omega^k S^k) \to \mathrm{end}(1_{k-1})$$

whose range consists entirely of equivalences.

In our previous exploration of these ideas [4], we emphasized the importance of a notion lying halfway between n-groupoids and fully general n-categories, which we called 'n-categories with duals'. The idea here is that duals are an interesting generalization of inverses. In particular, the 'tangle hypothesis' states that there is a 'free k-tuply monoidal n-category with duals on one object', $n\mathrm{Tang}_k$, having as n-morphisms certain n-dimensional surfaces in $[0,1]^{n+k}$ called 'framed n-tangles in $n+k$ dimensions'. For $n=1, k=2$ this was proved by Freyd and Yetter, Turaev, and Shum [30, 56, 65, 66]. This special case serves as the basis of recent work on 3-dimensional topological quantum field theory. Indeed, the fact that n-categories with duals are more general than n-groupoids is the reason why topological quantum field theory can give more refined information than homotopy theory. Recently progress has been made on the case $n=2$, $k=2$, which has also illuminated the theory of 2-braids in 4 dimensions [6, 7, 25, 43]. In general, we expect that in the stable range $n\mathrm{Tang}_k$ is equivalent to the stable n-category of 'framed cobordisms'. Also, the universal property of $n\mathrm{Tang}_k$ should give a k-tuply monoidal n-functor

$$T\colon n\mathrm{Tang}_k \to \Pi_n(\Omega^k S^k),$$

generalizing the Thom-Pontryagin construction. For more details the reader must turn to the references. Our main point here is that if the tangle hypothesis holds, a great deal, not only of homotopy theory, but also of topological quantum field theory arises naturally from the study of categorified analogs of $\mathbb{Z}$!

To conclude, let us note that all the entries in Table 4 should be equipped with 'multiplication' as well as 'addition' operations. For example, by virtue of being the free monoid on one element, $\mathbb{N}$ automatically becomes a rig in the following way: given an element $n \in \mathbb{N}$, there is a unique monoid homomorphism $f\colon \mathbb{N} \to \mathbb{N}$ with $f(1) = n$, namely multiplication by n. Likewise, by virtue of being the free group on one element, $\mathbb{Z}$ automatically becomes a ring. Categorifying once and stabilizing various numbers of times, we see that for $k > 0$, $1\mathrm{Braid}_k$ is a rig category and $\Pi_1(\Omega^k S^k)$ is a 'ring category': a rig category for which objects have additive inverses. This pattern should continue throughout the rest of Table 4.

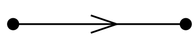
For example, in stable homotopy theory it is well known that the sphere spectrum is a 'ring spectrum' [51]. We expect that the sphere spectrum corresponds to the 'free $\mathbb{Z}$-groupoid on one object'. Indeed, Joyal has called the sphere spectrum 'the true integers', since it is an infinitely categorified, infinitely stabilized analog of $\mathbb{Z}$.

5 n-Categories and the Algebra of Opetopes

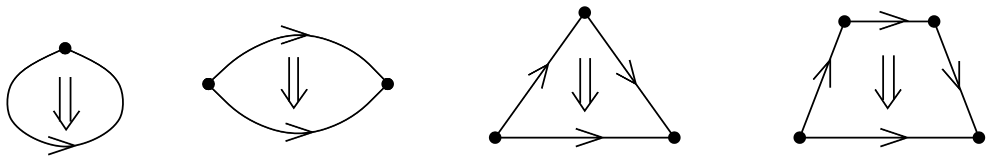
We hope the previous sections have whetted the reader's appetite for a rigorous theory of n-categories by sketching a bit of what we might do if we had one. Now we turn to the issue of actually developing this theory. It seems that any definition of n-category involves a choice of the basic shapes of j-morphisms — globes, simplices, or whatever. It also involves a choice of ways to compose j-morphisms by gluing these basic shapes together. Most importantly, it requires a careful treatment of coherence laws. In what follows we present an approach in which all these issues are handled simultaneously using the formalism of operads. In this approach, the basic shapes of j-morphisms are the j-dimensional 'opetopes'. The allowed ways of composing j-morphisms correspond precisely to the $(j + 1)$-dimensional opetopes. Moreover, the sequence of higher coherence laws satisfied by composition correspond to opetopes of ever higher dimension.

Before going into the details, let us sketch how this works. First consider some low-dimensional opetopes. The only 0-dimensional opetope is the point:

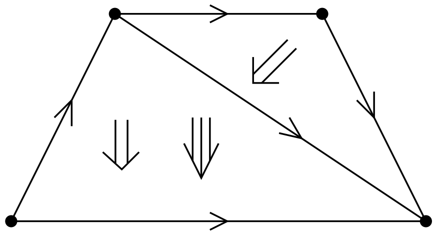
The only way to glue together 0-dimensional opetopes is the trivial way: the identity operation. The only 1-dimensional opetope is thus the interval, or more precisely the arrow:



The allowed ways of gluing together 1-dimensional opetopes are given by the 2-dimensional opetopes. The first few 2-dimensional opetopes are as follows:

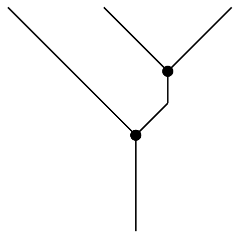


The allowed ways of gluing together 2-dimensional opetopes are given by the 3-dimensional opetopes. There are many of these; a simple example is as follows:



This may be a bit hard to visualize, but it depicts a 3-dimensional shape whose front consists of two 3-sided 'infaces', and whose back consists of a single 4-sided 'outface'. We have drawn double arrows on the infaces but not on the outface. Note that while this shape is topologically a ball, it cannot be realized as a polyhedron with planar faces. This is typical of opetopes.

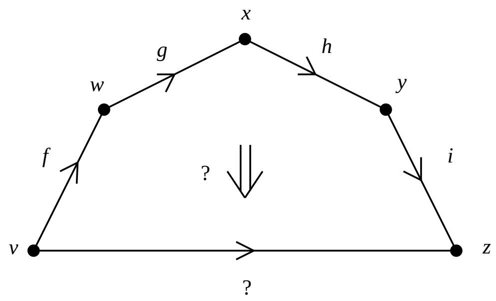
In general, an $(n+1)$-dimensional opetope has any number of infaces and exactly one outface: the infaces are n-dimensional opetopes glued together in a tree-like pattern, while the outface is a single n-dimensional opetope. For example, the 3-dimensional opetope above corresponds to the following tree:



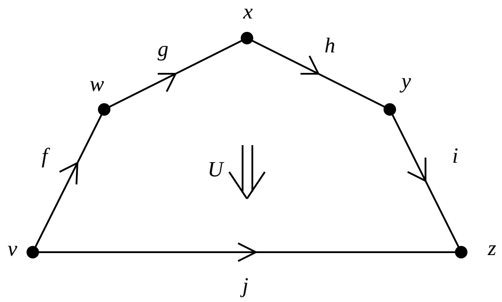
The two triangular infaces of the opetope correspond to the two nodes in this tree. This is a rather special tree; in general, we allow nonplanar trees with any number of nodes and any number of edges coming into each node.

Our approach to n-categories is a bit like the Kan complex approach to n-groupoids described in Section 3, but with simplicial sets replaced by 'opetopic sets'. Basically, an opetopic set is a set of 'cells' shaped like opetopes, such that any face of a cell is again a cell. In an n-category, the j-dimensional cells play the role of j-morphisms. An opetopic set is an n-category if it satisfies the following two properties:

1) *"Any niche has a universal occupant."* A 'niche' is a configuration where all the infaces of an opetope have been filled in by cells, but not the outface or the opetope itself:

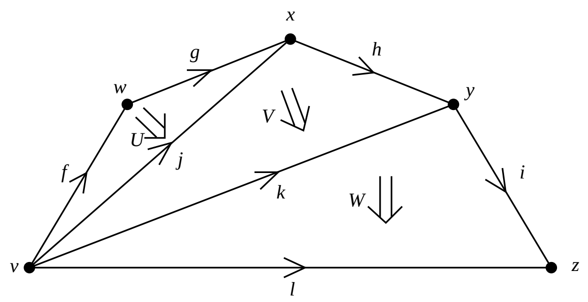


An 'occupant' of the niche is a way of extending this configuration by filling in the opetope (and thus its ouface) with a cell:



The 'universality' of an occupant means roughly that every other occupant factors through the given one *up to equivalence.* To make this precise we need to define universality in a rather subtle recursive way. We may think of a universal occupant of a niche as 'a process of composing' the infaces, and its outface as 'a composite' of the infaces.

2) *"Composites of universal cells are universal."* Suppose that U, V, and W below are universal cells:



Then we can compose them, and we are guaranteed that their composite is again universal, and thus that the outface l is a composite of the cells f, g, h, i. Note that a process of composing U, V, W is described by a universal occupant of a niche of one higher dimension.

Note that in this approach to n-categories, composition of cells is not an operation in the traditional sense: the composite is defined by a universal property, and is thus unique only up to equivalence. Only at the top level, for the n-cells of an n-category, is the composite truly unique. The main advantage of defining composition by a universal property is that we do not need to list coherence laws: all the right coherence laws arise automatically! This is a very important point, because in some sense it answers the puzzle concerning the origin of coherence laws.

At first this answer may seem as puzzling as the puzzle it answers. *Why* does defining composition by a universal property automatically generate all the right coherence laws? One reason is that coherence laws are 'right' when they hold in interesting examples, and in these examples composition is usually defined by a universal property. Consider for example the categorified version of $\mathbb{N}$ discussed in the Introduction: the category FinSet. Corresponding to addition in $\mathbb{N}$, the category FinSet has finite coproducts, i.e., disjoint unions. Coproducts are defined by a universal property, and this universal property immediately implies number of things. First, coproducts are unique up to canonical isomorphism. Second, if we pick a coproduct $x \sqcup y$ for every pair of objects $x, y \in \mathrm{FinSet}$, making disjoint union into an operation in the traditional sense, we obtain natural isomorphisms

$$a_{x,y,z} \colon (x \sqcup y) \sqcup z \to x \sqcup (y \sqcup z),$$

$$l_x \colon \emptyset \sqcup x \to x, \qquad r_x \colon x \sqcup \emptyset \to x.$$

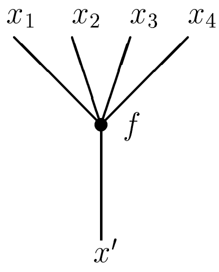
Third, these natural isomorphisms satisfy coherence laws making FinSet into a monoidal category. In short, the standard definition of monoidal category, which lists coherence laws, is best regarded as a spinoff of the fact that monoidal structures defined by universal properties automatically satisfy these laws.

Studying this example also suggests another idea which is built into our definition of n-categories. To prove the existence of the associator $a_{x,y,z}$ one uses the universal property for the coproduct of *three* objects, and to prove the existence of l_x and r_x one uses the universal property for the coproduct of *one* object. This suggests that in an n-category, arbitrary ℓ-ary composites should be treated on an equal footing with binary composites. The formalism of operads is admirably suited for this task.

In what follows we first review the theory of typed operads. Then we use this to define the opetopes, and more generally, 'O-opetopes' for any typed operad O. After a brief discussion of some notions concerning O-opetopic sets, we give the the definition of n-categories, and more generally, n-coherent O-algebras. We skim over some technical details which can be found in our paper [5].

5.1 Typed Operads

To describe 'many-sorted' or 'typed' algebraic structures using operads, we need to generalize the concept of operad a bit. For any set S of 'types', there is a notion of 'S-operad'. The basic idea is that for any $x_1, \ldots, x_\ell, x' \in S$, an S-operad O has a set $O(x_1, \ldots, x_\ell; x')$ of k-ary operations with inputs of type $x_1, \ldots, x_\ell$ and output of type x'. As in an ordinary untyped operad, we can visualize such an operation as a tree, but now we label the edges of the tree by types. For example, an operation $f \in O(x_1, \ldots, x_4; x')$ is drawn as follows:



More precisely:

Definition 1. *For any set S, an 'S-operad' O consists of*

1. *for any $x_1, \ldots, x_\ell, x' \in S$, a set $O(x_1, \ldots, x_\ell; x')$*
2. *for any $f \in O(x_1, \ldots, x_\ell; x')$ and any $g_1 \in O(x_{11}, \ldots, x_{1i_1}; x_1)$, $\ldots$, $g_\ell \in O(x_{\ell 1}, \ldots, x_{\ell i_\ell}; x_\ell)$, an element*
$$f \cdot (g_1, \ldots, g_\ell) \in O(x_{11}, \ldots, x_{1i_1}, \ldots\ldots, x_{\ell 1}, \ldots, x_{\ell i_\ell}; x')$$
3. *for any $x \in S$, an element $1_x \in O(x; x)$*
4. *for any permutation $\sigma \in S_\ell$, a map*
$$\begin{array}{rcl} \sigma\colon O(x_1, \ldots, x_\ell; x') & \to & O(x_{\sigma(1)}, \ldots, x_{\sigma(\ell)}; x') \\ f & \mapsto & f\sigma \end{array}$$

such that:

(a) *whenever both sides make sense,*
$$f \cdot (g_1 \cdot (h_{11}, \ldots, h_{1i_1}), \ldots, g_\ell \cdot (h_{\ell 1}, \ldots, h_{\ell i_\ell})) =$$
$$(f \cdot (g_1, \ldots g_\ell)) \cdot (h_{11}, \ldots, h_{1i_1}, \ldots\ldots, h_{\ell 1}, \ldots, h_{\ell i_\ell})$$

(b) *for any $f \in O(x_1, \ldots, x_\ell; x')$,*
$$f = 1_{x'} \cdot f = f \cdot (1_{x_1}, \ldots, 1_{x_\ell})$$

(c) *for any $f \in O(x_1, \ldots, x_\ell; x')$ and $\sigma, \sigma' \in S_\ell$,*
$$f(\sigma\sigma') = (f\sigma)\sigma'$$

(d) *for any* $f \in O(x_1, \ldots, x_\ell; x')$, $\sigma \in S_\ell$, *and* $g_1 \in O(x_{11}, \ldots, x_{1i_1}; x_1)$, $\ldots, g_\ell \in O(x_{\ell 1}, \ldots, x_{\ell i_\ell}; x_\ell)$,

$$(f\sigma) \cdot (g_{\sigma(1)}, \ldots, g_{\sigma(\ell)}) = (f \cdot (g_1, \ldots, g_\ell))\, \rho(\sigma),$$

where $\rho\colon S_\ell \to S_{i_1 + \cdots + i_\ell}$ *is the obvious homomorphism.*

(e) *for any* $f \in O(x_1, \ldots, x_\ell; x')$, $g_1 \in O(x_{11}, \ldots, x_{1i_1}; x_1), \ldots,$ $g_\ell \in O(x_{k1}, \ldots, x_{\ell i_\ell}; x_\ell)$, *and* $\sigma_1 \in S_{i_1}, \ldots, \sigma_\ell \in S_{i_\ell}$,

$$(f \cdot (g_1\sigma_1, \ldots, g_\ell\sigma_\ell)) = (f \cdot (g_1, \ldots, g_\ell))\, \rho'(\sigma_1, \ldots, \sigma_\ell),$$

where $\rho'\colon S_{i_1} \times \cdots \times S_{i_\ell} \to S_{i_1 + \cdots + i_\ell}$ *is the obvious homomorphism.*

There is an obvious notion of a morphism from an S-operad O to an S-operad O': a function mapping each set $O(x_1, \ldots, x_\ell; x')$ to the corresponding set $O'(x_1, \ldots, x_\ell; x')$, preserving composition, identities, and the symmetric group actions. An important example is an 'algebra' of an S-operad:

Definition 2. *For any S-operad O, an 'O-algebra' A consists of:*

1. *for any $x \in S$, a set $A(x)$.*
2. *for any $f \in O(x_1, \ldots, x_\ell; x')$, a function*

$$\alpha(f)\colon A(x_1) \times \cdots \times A(x_\ell) \to A(x')$$

such that:

(a) *whenever both sides make sense,*

$$\alpha(f \cdot (g_1, \ldots, g_\ell)) = \alpha(f)(\alpha(g_1) \times \cdots \times \alpha(g_\ell))$$

(b) *for any $x \in C$, $\alpha(1_x)$ acts as the identity on $A(x)$*

(c) *for any $f \in O(x_1, \ldots, x_\ell, x')$ and $\sigma \in S_\ell$,*

$$\alpha(f\sigma) = \alpha(f)\sigma,$$

where $\sigma \in S_\ell$ acts on the function $\alpha(f)$ on the right by permuting its arguments.

In what follows, by 'operad' we will mean an S-operad for some set S of types. We can think of such an operad as a simple sort of theory, and its algebras as models of this theory. Thus we can study operads either 'syntactically' or 'semantically'. To describe an operad syntactically, we list:

1. the set S of *types*,
2. the sets $O(x_1, \ldots, x_\ell; x')$ of *operations*,
3. the set of all *reduction laws* saying that some composite of operations (possibly with arguments permuted) equals some other operation.

This is like a presentation in terms of generators and relations, with the reduction laws playing the role of relations. On the other hand, to describe an operad semantically, we describe its algebras.

For example, the simplest operad is the 'initial untyped operad' I. Syntactically, this is the S-operad with:

1. only one type: $S = \{x\}$,
2. only one operation, the identity operation $1 \in O(x; x)$,
3. all possible reduction laws.

Semantically, I is the operad whose algebras are just sets.

Another important operad is the 'terminal untyped operad' T. This is the S-operad with

1. only one type: $S = \{x\}$,
2. exactly one operation of each arity,
3. all possible reduction laws.

The algebras of T are commutative monoids, with the ℓ-ary operation being ℓ-fold multiplication, or the unit element when $\ell = 0$, since nullary operations correspond to 'constants'.

5.2 Opetopes

The following fact is the key to defining the opetopes. Let O be an S-operad, and let $\mathrm{elt}(O)$ be the set of all operations of O.

Theorem 3. *There is an* $\mathrm{elt}(O)$*-operad* O^+ *whose algebras are* S*-operads over* O*, i.e.,* S*-operads equipped with a homomorphism to* O*.*

We call O^+ the 'slice operad' of O. One can describe O^+ syntactically as follows:

1. The types of O^+ are the operations of O.
2. The operations of O^+ are the reduction laws of O.
3. The reduction laws of O^+ are the ways of combining reduction laws of O to give other reduction laws.

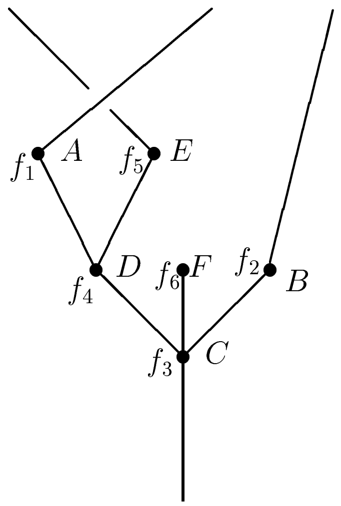
The 'level-shifting' going on here as we pass from O to O^+ is a way of systematizing the process of categorification.

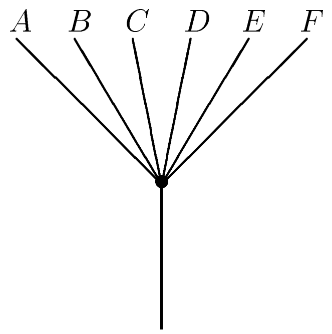
A nice example of the slice operad construction is the operad I^+. The algebras of this operad turn out to be monoids. Another nice example is the operad T^+, whose algebras are untyped operads! It is also very interesting to iterate the slice operad construction. For example, the algebras of I^{++} work out to be exactly untyped planar operads. More generally, let O^{n+} be the operad formed by applying the slice operad construction n times to the operad O, or just O itself if $n = 0$.

Definition 4. *An* n*-dimensional '*O*-opetope' is a type of* O^{n+}*, or equivalently, if* $n \geq 1$*, an operation of* $O^{(n-1)+}$*.*

In particular, we define an n-dimensional 'opetope' to be an n-dimensional O-opetope for $O = I$, the initial untyped operad. The 0-dimensional opetopes are thus the types of I, but there is only one type, so there is only one 0-dimensional opetope, which we visualize as a point. The 1-dimensional opetopes are the types of I^+, or in other words, the operations of I. I has only one operation, the identity, so there is only one 1-dimensional opetope, which we visualize as an interval. The 2-dimensional opetopes are the types of I^{++}, or in other words, the operations of I^+, which are the reduction laws of I. These reduction laws all state that the identity operation composed with itself ℓ times equals itself. This leads to 2-dimensional opetopes with ℓ infaces and one outface. Actually there are $\ell!$ different 2-dimensional opetopes with ℓ infaces, since the permutation group S_ℓ acts freely on the set of ℓ-ary operations of I^+. We could keep track of these by labelling the infaces with some permutation of ℓ distinct symbols. A more systematic approach is to use 'metatree notation'. In this notation, any n-dimensional O-opetope is represented as a list of n labelled trees.

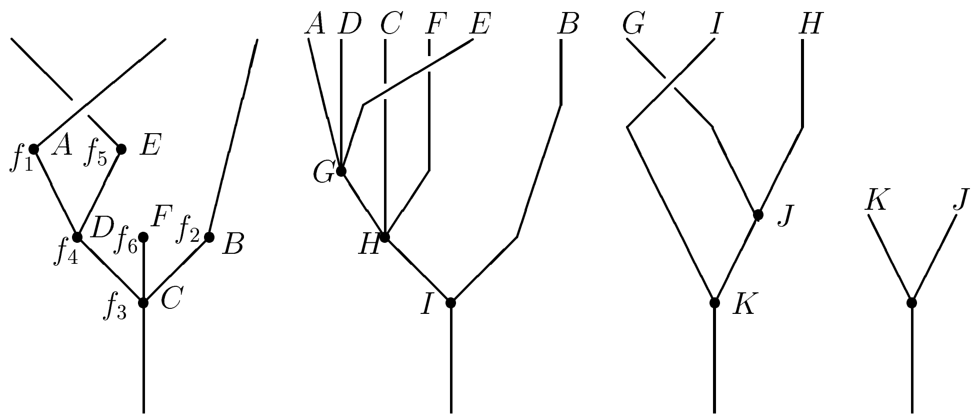
To see how this works, first consider the 2-dimensional O-opetopes, which are the operations of O^+. An operation of O^+ can be specified as in the figure below.





The first tree is an arbitrary 'O-tree'. This is a combed tree (i.e., planar except for a permutation of inputs at the top) with nodes labelled by operations of O. We require that a node labelled by a ℓ-ary operation have ℓ edges coming into it from above. Moreover, we require that it be possible to label every edge with an element of S in such a way that for any node labelled by an operation in $O(x_1, \ldots, x_\ell; x')$, the edges coming into that node from above are labelled by the types $x_1, \ldots, x_\ell$ in that order, while the edge coming out of it from below is labelled by the type x'. We also label each node of this first tree with a distinct symbol A, B, C, etc.. The second tree is planar and has only one node, with n edges coming into that node from above, labelled by the same symbols $A, B, C, \ldots$ in any order. These specify the order of the input types of the operation of O^+ we are describing.

More generally, for any $n > 1$ one can specify any n-dimensional O-opetope by means of an 'n-dimensional metatree'. Here is an example for $n = 3$:



An n-dimensional metatree is a list of n labelled trees, the last of which is a planar tree with only one node, while the rest are combed trees. The first tree is an arbitrary O-tree. For $1 \le i < n$, every node of the ith tree is labelled with a distinct symbol, and the same symbols also label all the edges at the very top of the $(i+1)$st tree, each symbol labelling exactly one edge. In addition, each edge of the $(i+1)$st tree must correspond to a subtree of the ith tree in such a way that:

1. The edge at the very top of the $(i+1)$st tree labelled by a given symbol corresponds to the subtree of the ith tree whose one and only node is labelled by the same symbol.

2. The edge of the $(i+1)$st tree coming out of a given note from below corresponds to the subtree that is the union of the subtrees corresponding to the edges coming into that node from above.

3. The edge at the very bottom of the $(i+1)$st tree corresponds to the whole ith tree.

Special care must be taken when the node of the last tree has no edges coming into it from above. This can only occur when all the previous trees are empty. This sort of metatree describes a nullary operation of $O^{(n-1)+}$ whose output type is an identity operation 1_x of $O^{(n-2)+}$. To specify which identity operation, we need to label the edge coming out of the node of the last tree from below with the operation 1_x.

6 n-Coherent operad algebras

An 'n-coherent O-algebra' is an n times categorified analog of an algebra of the operad O. In particular, when $O = I$, an n-coherent O-algebra is just an n-category, which is the n times categorified analog of a set. An n-coherent O-algebra is an 'O-opetopic set' with certain properties. We omit the precise definition of O-opetopic sets here. For our purposes, it should suffice to know that an O-opetopic set is very much like a simplicial set, but with O-opetopes replacing simplices, and no 'degeneracy maps', only 'face maps'. An O-opetopic set thus consists of (possibly empty) collections of 'cells' shaped like all the different O-opetopes, such that any inface or outface of a cell is again a cell.

If $j \geq 1$, we may schematically represent a j-dimensional cell x in an O-opetopic set as follows:

$$(a_1, \ldots, a_\ell) \xrightarrow{\quad x \quad} a'$$

Here $a_1, \ldots, a_\ell$ are the infaces of x and a' is the outface of x; all these are cells of one lower dimension. A configuration just like this, satisfying all the incidence relations satisfied by the boundary of a cell, but with x itself missing:

$$(a_1, \ldots, a_\ell) \xrightarrow{\quad ? \quad} a'$$

is called a 'frame'. A 'niche' is like a frame with the outface missing:

$$(a_1, \ldots, a_\ell) \xrightarrow{\quad ? \quad} ?$$

Similarly, a 'punctured niche' is like a frame with the outface and one inface missing:

$$(a_1, \ldots, a_{i-1}, ?, a_{i+1}, \ldots, a_\ell) \xrightarrow{\quad ? \quad} ?$$

If one of these configurations (frame, niche, or punctured niche) can be extended to an actual cell, the cell is called an 'occupant' of the configuration. Occupants of the same frame are called 'frame-competitors', while occupants of the same niche are called 'niche-competitors'.

Next we need the concept of a 'universal occupant' of a niche. Since the definition of this concept looks rather formidable at first, we first give a heuristic explanation. As already noted, the main use of universality to define composites:

Definition 5. *Given a universal occupant u of a j-dimensional niche:*

$$(a_1, \ldots, a_k) \xrightarrow{\quad u \quad} b$$

we call b a 'composite' of $(a_1, \ldots, a_k)$.

To understand universality more deeply, one must understand the role played by cells of different dimensions. In our framework an n-category usually has cells of arbitrarily high dimension, just like a Kan complex. For $j \leq n$ the j-dimensional cells play the role of j-morphisms, while for $j > n$ they play the role of 'equations', 'equations between equations', and so on. The definition of universality depends on n in a way that has the following effects. For $j \leq n$ there may be many universal occupants of a given j-dimensional niche, which is why we speak of 'a' composite rather than 'the' composite. There is at most one occupant of any given $(n+1)$-dimensional niche, which is automatically universal. Thus composites of n-cells are unique, and we may think of the universal occupant of an $(n+1)$-dimensional niche as an equation saying that the composite of the infaces equals the outface. For $j > n+1$ there is exactly one occupant of each j-dimensional frame, indicating that the composite of the equations corresponding to the infaces equals the equation corresponding to the outface.

The definition of universality essentially says that a j-dimensional niche-occupant is universal if all of its niche-competitors factor through it uniquely, *up to equivalence*. For $j \geq n+1$ this amounts to saying that each niche has a unique occupant, while for $j = n$ it means that each niche has an occupant through which all of its niche-competitors factor uniquely. Technically, the definition of universality says that composition with a universal niche-occupant set up a 'balanced punctured niche' of one higher dimension. One should think of a balanced punctured niche as defining an equivalence between occupants of its outface and occupants of its missing outface.

Now let us give the actual definition:

Definition 6. *A j-dimensional niche-occupant:*

$$(c_1, \ldots, c_k) \xrightarrow{\quad u \quad} d$$

is said to be 'universal' if and only if $j > n$ and u is the only occupant of its niche, or $j \leq n$ and for any frame-competitor d' of d, the $(j+1)$-dimensional punctured niche

$$\begin{tikzcd} ((c_1, \ldots, c_k) \xrightarrow{u} d,\ d \xrightarrow{?} d') \arrow[d, "?"] \\ (c_1, \ldots, c_k) \xrightarrow{?} d' \end{tikzcd}$$

and its mirror-image version

$$\begin{tikzcd} (d \xrightarrow{?} d',\ (c_1, \ldots, c_k) \xrightarrow{u} d) \arrow[d, "?"] \\ (c_1, \ldots, c_k) \xrightarrow{?} d' \end{tikzcd}$$

are balanced.

Of course, now we need the definition of 'balanced'. The reader will note that while the definitions of 'universal' and 'balanced' call upon each other, there is no bad circularity.

Definition 7. *An m-dimensional punctured niche:*

$$(a_1, \ldots, a_{i-1}, ?, a_{i+1}, \ldots, a_k) \xrightarrow{\quad ? \quad} ?$$

is said to be 'balanced' if and only if $m > n + 1$ or:

1. *any extension*

$$(a_1, \ldots, a_{i-1}, ?, a_{i+1}, \ldots, a_k) \xrightarrow{\quad ? \quad} b$$

extends further to:

$$(a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_k) \xrightarrow{\quad u \quad} b$$

with u universal in its niche, and

2. *for any occupant*

$$(a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_k) \xrightarrow{\quad u \quad} b$$

universal in its niche, and frame-competitor a'_i *of* a_i*, the* $(m+1)$*-dimensional punctured niche*

$$\begin{array}{c} (a'_i \xrightarrow{?} a_i,\ (a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_k) \xrightarrow{u} b) \\ \Big\downarrow {\scriptstyle ?} \\ (a_1, \ldots, a_{i-1}, a'_i, a_{i+1}, \ldots, a_k) \xrightarrow{?} b \end{array}$$

and its mirror-image version

$$\begin{array}{c} ((a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_k) \xrightarrow{u} b,\ a'_i \xrightarrow{?} a_i) \\ \Big\downarrow {\scriptstyle ?} \\ (a_1, \ldots, a_{i-1}, a'_i, a_{i+1}, \ldots, a_k) \xrightarrow{?} b \end{array}$$

are balanced.

Note that the first numbered condition in the definition of 'balanced' definition generalizes the concept of an essentially surjective functor, while the second generalizes the concept of a fully faithful functor.

Finally, we define n-coherent O-algebras and various special cases:

Definition 8. *An 'n-coherent O-algebra' is an O-opetopic set such that 1) every niche has a universal occupant, and 2) composites of universal cells are universal.*

Definition 9. *An 'n-category' is an n-coherent I-algebra.*

Definition 10. *A 'monoidal n-category' is an n-coherent I^+-algebra.*

Definition 11. *A 'stable n-category' is an n-coherent T-algebra.*

One can show that any S-operad morphism $f\colon O \to O'$ allows one to turn an n-coherent O'-algebra A into an n-coherent O-algebra f^*A. Thus any stable n-category has an underlying monoidal n-category, and any monoidal n-category has an underlying n-category.

Given an n-category with 0-cells x and y, there is an $(n-1)$-category $\hom(x, y)$. One can also construct a stable $(n+1)$-category of all (small) n-categories, though the details of this construction have not yet been published. Using these facts, one can give rigorous formulations of many processes going between neighboring entries in Table 2: decategorification, discrete categorification, delooping, looping, forgetting monoidal structure, stabilization, and the generalized center construction. However, it remains to make precise and prove the stabilization hypothesis in this framework. Basically, one wishes to show that for $k \geq n+2$, the $(n+1)$-category of all stable n-categories is equivalent to the full sub-$(n+1)$-category of all n-categories having only one 0-cell and only one j-cell in each frame for $0 < j < k$.

7 Conclusions

In this paper we have discussed iterated categorifications and stabilizations of some of the very simplest algebraic structures: the natural numbers and the integers. However, one can also categorify many other concepts: vector spaces [40] and Hilbert spaces [2], group algebras [21], algebras of formal power series [5, 37] and other Hopf algebras [20, 22], sheaves [15, 17], and so on. Interesting results about these familiar structures typically have interesting categorified analogs. It is clear, therefore, that the set-based mathematics we know and love is just the tip of an immense iceberg of n-categorical, and ultimately ω-categorical, mathematics.

The prospect of exploring this huge body of new mathematics is both exhilarating and daunting. The basic philosophy is simple: *never mistake equivalence for equality*. The technical details, however, are not so simple — at least not yet. To proceed efficiently it is crucial that we gain a clearer understanding of the foundations before rushing ahead with complicated constructions.

Many basic questions remain open. For example, how significant is the fact that operads play a role both in the theory of E_k spaces and the definition of n-categories described above? Operads are very versatile, so this might at first seem to be a coincidence. However, there are deep relationships between operads, categorification, and the theory of algebraic structures satisfying laws 'up to coherent homotopy' [60]. In particular, Trimble [64] has pointed out an interesting connection. For many purposes it is best to think of E_k spaces as algebras, not of the little k-cubes operad, but of a closely related operad $F(k)$ discovered by Getzler and Jones [33]. The space $F(k)_\ell$ is the Fulton-MacPherson compactification of the configuration space of ℓ points in $\mathbb{R}^k$ modulo translations and dilations [31]. In particular, $F(1)_\ell$ is just $K_\ell \times S_\ell$, where K_ℓ is the $(\ell - 2)$-dimensional associahedron.

Since homotopy n-types of E_1 spaces correspond to monoidal n-groupoids, while n-coherent I^+-algebras are monoidal n-categories, one might expect a relationship between $F(1)$ and the I^+-opetopes. The associahedron K_ℓ has a cell decomposition having cells in one-to-one correspondence with planar trees with ℓ leaves for which all nodes have at least one edge coming in from above. It follows that the cells in the corresponding decomposition of $F(1)$ correspond to a certain class of 2-dimensional I^+-opetopes, or equivalently, 3-dimensional opetopes. Is there a deeper relation between opetopes and the associahedron? This might shed new light on the origin of coherence laws.

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