Categorification and Topology

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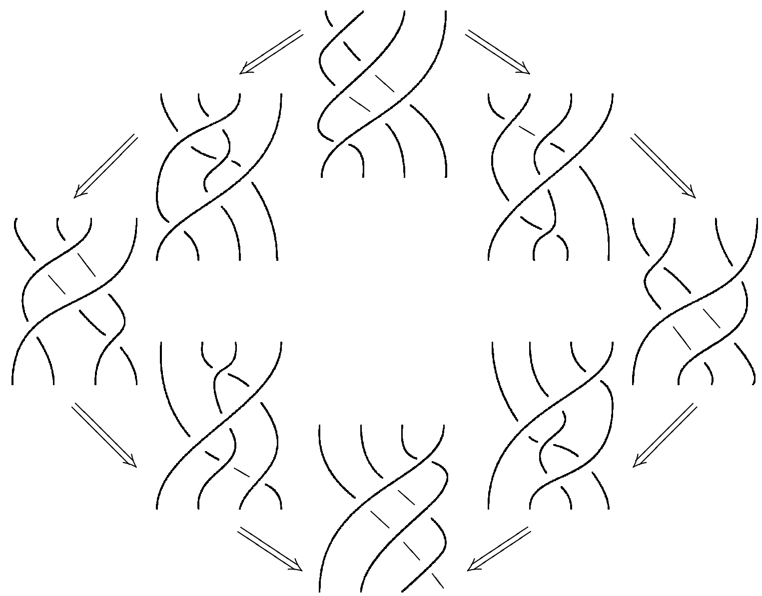
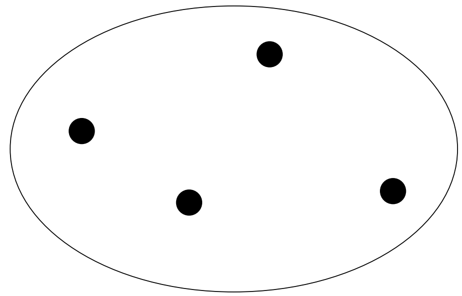


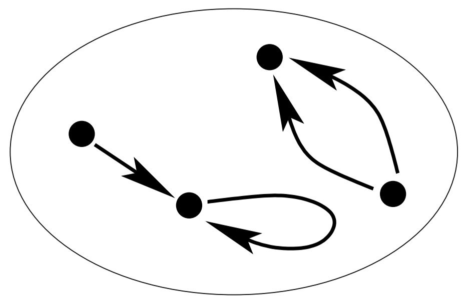
for references and more, see:

http://math.ucr.edu/home/baez/cat/

Once upon a time, mathematics was all about *sets*:

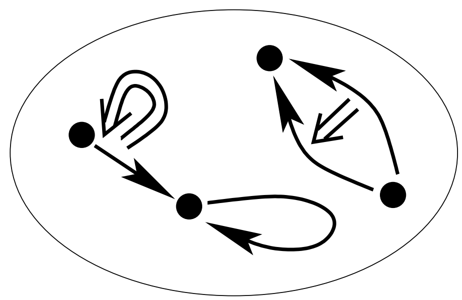


In 1945, Eilenberg and Mac Lane introduced *categories*:



Category theory puts *processes* (morphisms): $\bullet \rightarrow \bullet$
on an equal footing with *things* (objects): $\bullet$

In 1967 Bénabou introduced *weak 2-categories*:



These include *processes between processes*, or '2-morphisms':

$$\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \bullet \end{tikzcd}$$

We can compose 2-morphisms vertically:

$$
\begin{tikzcd}[column sep=huge]
x \arrow[r, bend left=60, "f"{name=U}] \arrow[r, "f'"{name=M}] \arrow[r, bend right=60, "f''"'{name=D}] \arrow[Rightarrow, from=U, to=M, shorten=2pt, "\alpha"] \arrow[Rightarrow, from=M, to=D, shorten=2pt, "\alpha'"] & y
\end{tikzcd}
\;=\;
\begin{tikzcd}[column sep=huge]
x \arrow[r, bend left=40, "f"{name=U}] \arrow[r, bend right=40, "f''"'{name=D}] \arrow[Rightarrow, from=U, to=D, "\alpha\alpha'"] & z
\end{tikzcd}
$$

or horizontally:

$$
\begin{tikzcd}
x \arrow[r, bend left, "f"{name=U1}] \arrow[r, bend right, "f'"'{name=D1}] \arrow[Rightarrow, from=U1, to=D1, "\alpha"'] & \bullet \arrow[r, bend left, "g"{name=U2}] \arrow[r, bend right, "g'"'{name=D2}] \arrow[Rightarrow, from=U2, to=D2, "\beta"'] & z
\end{tikzcd}
\;=\;
\begin{tikzcd}[column sep=huge]
x \arrow[r, bend left=40, "fg"{name=U}] \arrow[r, bend right=40, "f'g'"'{name=D}] \arrow[Rightarrow, from=U, to=D, "\alpha\otimes\beta"] & z
\end{tikzcd}
$$

and various laws hold, including the 'interchange' law:

$$
\begin{tikzcd}[column sep=huge]
\bullet \arrow[r, bend left=60, "f"{name=U1}] \arrow[r, "f'"{name=M1}] \arrow[r, bend right=60, "f''"'{name=D1}] \arrow[Rightarrow, from=U1, to=M1, shorten=2pt, "\alpha"] \arrow[Rightarrow, from=M1, to=D1, shorten=2pt, "\alpha'"] & \bullet \arrow[r, bend left=60, "g"{name=U2}] \arrow[r, "g'"{name=M2}] \arrow[r, bend right=60, "g''"'{name=D2}] \arrow[Rightarrow, from=U2, to=M2, shorten=2pt, "\beta"] \arrow[Rightarrow, from=M2, to=D2, shorten=2pt, "\beta'"] & \bullet
\end{tikzcd}
$$

$$
\boldsymbol{(\alpha\alpha') \otimes (\beta\beta') = (\alpha \otimes \beta)(\alpha' \otimes \beta')}
$$

We call these 2-categories 'weak' because all laws between morphisms hold only *up to 2-isomorphisms*, which satisfy laws of their own.

For example, we have an 'associator'

$$a_{f,g,h} \colon (fg)h \Rightarrow f(gh)$$

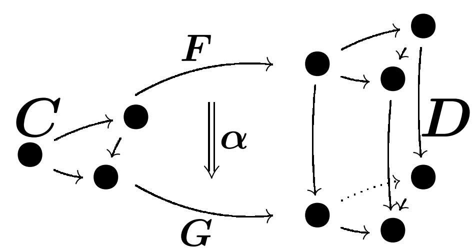
which satisfies the 'pentagon equation':

$$\begin{tikzcd}
 & & (fg)(hi) \arrow[drr, Rightarrow, "a_{f,g,hi}"] & & \\
((fg)h)i \arrow[urr, Rightarrow, "a_{fg,h,i}"] \arrow[dr, Rightarrow, "a_{f,g,h} \otimes 1_i"'] & & & & f(g(hi)) \\
 & (f(gh))i \arrow[rr, Rightarrow, "a_{f,gh,i}"'] & & f((gh)i) \arrow[ur, Rightarrow, "1_f \otimes a_{g,h,i}"'] &
\end{tikzcd}$$

The 'set of all sets' is really a category: Set.

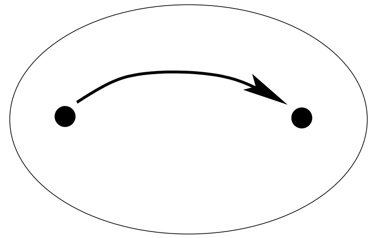
The 'category of all categories' is really a 2-category: Cat. It has:

- categories as objects,
- functors as morphisms,
- natural transformations as 2-morphisms.



Cat is a 'strict' 2-category: all laws hold *exactly*, not just up to isomorphism. But there are also many interesting *weak* 2-categories!

For example, any topological space has a 'fundamental groupoid':



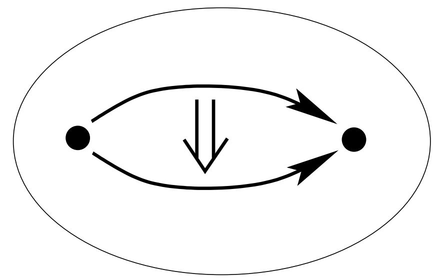
This is a category with:

- points as objects,
- homotopy classes of paths as morphisms.

All morphisms are invertible, so it's a 'groupoid'.

Indeed, groupoids are equivalent to 'homotopy 1-types': spaces X with $\pi_n(X, x)$ trivial for all $n > 1$ and all $x \in X$.

Any space also has a *fundamental 2-groupoid*:



This is a weak 2-category with:

- points as objects,
- paths as morphisms,
- homotopy classes of 'paths of paths' as 2-morphisms.

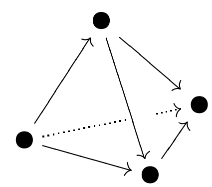
All 2-morphisms are invertible and all morphisms are weakly invertible, so it's a 'weak 2-groupoid'.

And indeed, weak 2-groupoids are equivalent to homotopy 2-types!

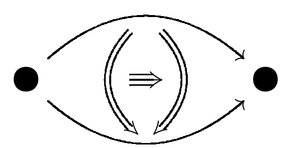
Around 1975, Grothendieck suggested:

The Homotopy Hypothesis: Weak n-groupoids are equivalent to homotopy n-types.

This was easy to prove using the simplicial approach to weak ∞-groupoids, where they are called 'Kan complexes':



It is not yet proved in the 'globular' approach:



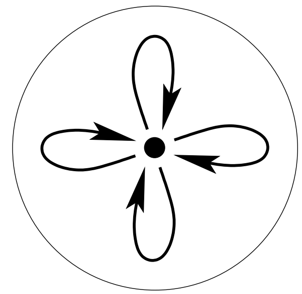
In 1995, Gordon, Power and Street introduced a globular approach to *weak 3-categories*.

In 1998, Batanin introduced globular *weak* ∞*-categories*.

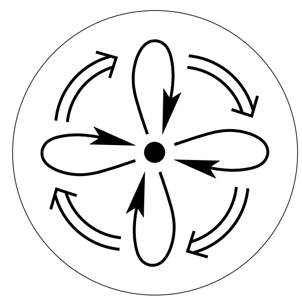
There are also many other approaches, and *simplicial* approaches have been the most successful in topology. To orient ourselves in this complicated field, we need some hypotheses about how n-categories work.

Many of these involve the 'Periodic Table'.

A category with one object is a 'monoid' — a set with associative multiplication and a unit element:



A 2-category with one object is a 'monoidal category' — a category with an associative tensor product: and a unit object:



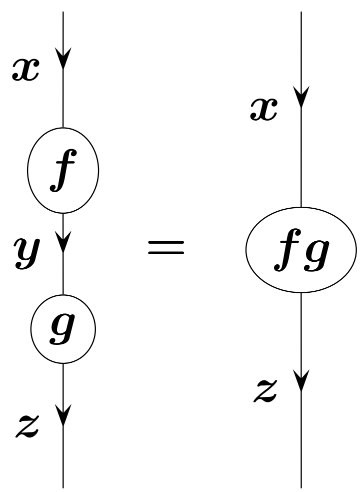
Now associativity and the unit laws are 'weakened':

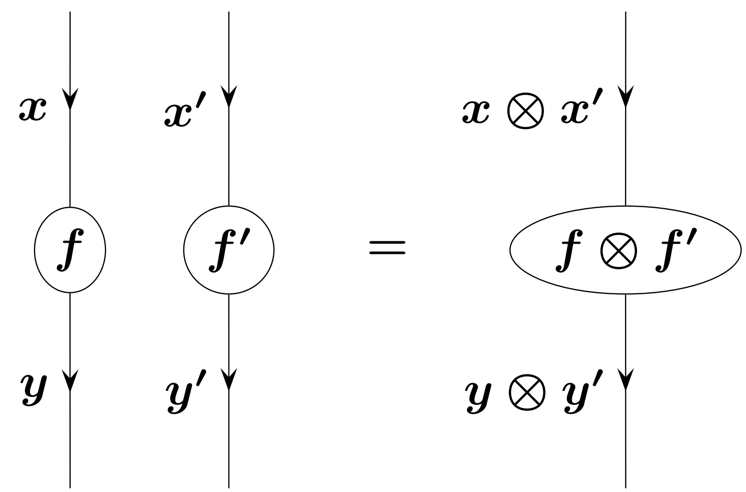
$$(\boldsymbol{x} \otimes \boldsymbol{y}) \otimes \boldsymbol{z} \cong \boldsymbol{x} \otimes (\boldsymbol{y} \otimes \boldsymbol{z}), \qquad \boldsymbol{I} \otimes \boldsymbol{x} \cong \boldsymbol{x} \cong \boldsymbol{x} \otimes \boldsymbol{I}$$

To regard a 2-category with one object as a monoidal category:

- **we ignore the object,**
- **we rename the morphisms 'objects',**
- **we rename the 2-morphisms 'morphisms'.**

Vertical and horizontal composition of 2-morphisms become composition and tensoring of morphisms:





In general, we may define an n-category with one object to be a 'monoidal $(n-1)$-category'.

For example:

- **Set is a monoidal category, using the cartesian product $S \times T$ of sets.**
- **Cat is a monoidal 2-category, using the cartesian product $C \times D$ of categories.**
- **We expect that nCat is a monoidal $(n+1)$-category!**

QUESTION: what's a *monoidal category* with just one object? It must be some sort of monoid...

It has one object, namely the unit I, and a set of morphisms $\alpha\colon I \to I$. We can compose morphisms:

$$\alpha\beta$$

and also tensor them:

$$\alpha \otimes \beta$$

Composition and tensoring are related by the interchange law:

$$(\alpha\alpha') \otimes (\beta\beta') = (\alpha \otimes \beta)(\alpha' \otimes \beta')$$

So, we can carry out the 'Eckmann–Hilton argument':

$$
\begin{array}{|c|c|}\hline \alpha & \beta \\ \hline \end{array}
\qquad
\begin{array}{|c|c|}\hline \alpha & 1 \\ \hline 1 & \beta \\ \hline \end{array}
\qquad
\begin{array}{|c|}\hline \alpha \\ \hline \beta \\ \hline \end{array}
\qquad
\begin{array}{|c|c|}\hline 1 & \alpha \\ \hline \beta & 1 \\ \hline \end{array}
\qquad
\begin{array}{|c|c|}\hline \beta & \alpha \\ \hline \end{array}
$$

$$
\begin{array}{ccccccc}
\alpha \otimes \beta & = (\alpha \otimes 1)(1 \otimes \beta) & & & (1\beta) \otimes (\alpha 1) & = & \beta \otimes \alpha \\
 & \| & & & \| & & \\
 & (\alpha 1) \otimes (1\beta) & = & \alpha\beta & = & (1 \otimes \alpha)(\beta \otimes 1) &
\end{array}
$$

ANSWER: a monoidal category with one object is a *commutative* monoid!

In other words: a 2-category with one object and one morphism is a commutative monoid.

What's the pattern?

An $(n+k)$-category with only one j-morphism for $j < k$ can be reinterpreted as an n-category.

But, it will be an n-category with k ways to 'multiply': a *k-tuply monoidal n-category*.

When there are several ways to multiply, the Eckmann–Hilton argument gives a kind of 'commutativity'.

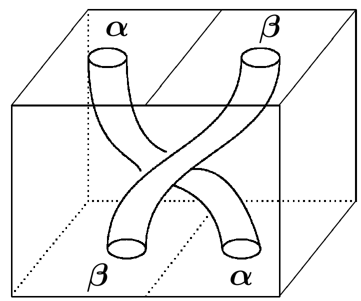
Our guesses are shown in the Periodic Table...

k-tuply monoidal n-categories

	$n = 0$	$n = 1$	$n = 2$
$k = 0$	sets	categories	2-categories
$k = 1$	monoids	monoidal categories	monoidal 2-categories
$k = 2$	commutative monoids	braided monoidal categories	braided monoidal 2-categories
$k = 3$	‘’	symmetric monoidal categories	sylleptic monoidal 2-categories
$k = 4$	‘’	‘’	symmetric monoidal 2-categories
$k = 5$	‘’	‘’	‘’

Consider $n = 1$, $k = 2$: a doubly monoidal 1-category is a *braided monoidal category*. The Eckmann–Hilton argument gives the braiding:

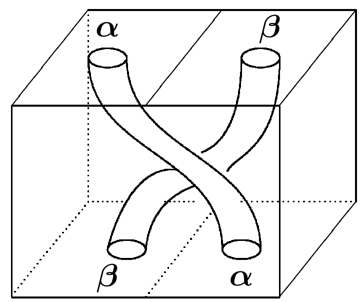
$$\begin{array}{|c|c|}\hline \alpha & \beta \\ \hline\end{array} \cong \begin{array}{|c|c|}\hline \alpha & 1 \\ \hline 1 & \beta \\ \hline\end{array} \cong \begin{array}{|c|}\hline \alpha \\ \hline \beta \\ \hline\end{array} \cong \begin{array}{|c|c|}\hline 1 & \alpha \\ \hline \beta & 1 \\ \hline\end{array} \cong \begin{array}{|c|c|}\hline \beta & \alpha \\ \hline\end{array}$$



$$B_{\alpha,\beta} \colon \alpha \otimes \beta \xrightarrow{\sim} \beta \otimes \alpha$$

The *process of proving an equation* has become an *isomorphism!* This happens when we move one step right in the Periodic Table.

Indeed, a *different proof* of commutativity becomes a *different isomorphism*:



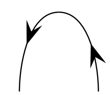
$$B^{-1}_{\beta,\alpha}\colon \alpha \otimes \beta \xrightarrow{\sim} \beta \otimes \alpha$$

This explains the existence of knots!

Shum's theorem: $1\mathrm{Tang}_2$, the category of 1d tangles in a (2+1)-dimensional cube, is the free braided monoidal category with duals on one object x: the positively oriented point.

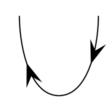
An object α in a monoidal category has a 'dual' α^* if there is a 'unit'

$$i_\alpha \colon I \to \alpha \otimes \alpha^*$$



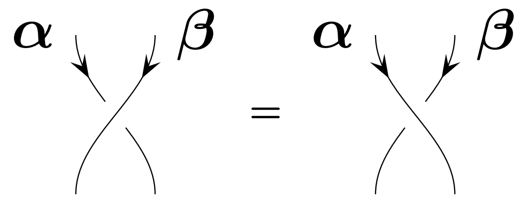
and 'counit'

$$e_\alpha \colon \alpha^* \otimes \alpha \to I$$



satisfying the 'zig-zag equations':

A triply monoidal 1-category is a *symmetric monoidal category*. Now we have 3 dimensions of space instead of just 2. This makes the two ways of moving α past β equal:



So, the situation is 'more commutative'. This happens when we move one step <u>down</u> in the Periodic Table.

We can untie all knots in 4d:

Theorem: $1\mathrm{Tang}_3$, the category of 1d tangles in a $(3+1)$-dimensional cube, is the free symmetric monoidal category with duals on one object.

However, as we march down any column of the Periodic Table, k-tuply monoidal n-categories seem to become 'maximally commutative' when k reaches $n + 2$.

For example, you can untie all n-dimensional knots in a $(2n + 2)$-dimensional cube. Extra dimensions don't help! The Freudenthal Suspension Theorem is another big piece of evidence: the homotopy n-type of a k-fold loop space stabilizes when $k \geq n + 2$.

So, Larry Breen, James Dolan and I guessed:

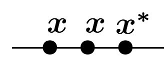
The Stabilization Hypothesis: k-tuply monoidal n-categories are equivalent to $(k + 1)$-tuply monoidal n-categories when $k \geq n + 2$.

Let us call these *stable n-categories.*

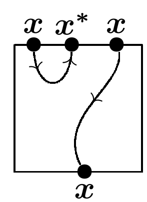
Now let's look harder at higher-dimensional knot theory. Consider $n\mathrm{Tang}_k$, the n-category of n-dimensional tangles in a $(k+n)$-dimensional cube.

For example $2\mathrm{Tang}_1$ has:

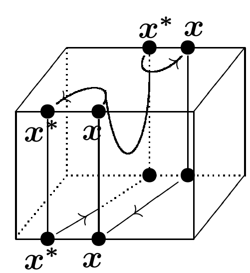
- collections of oriented points in the 1-cube as objects:



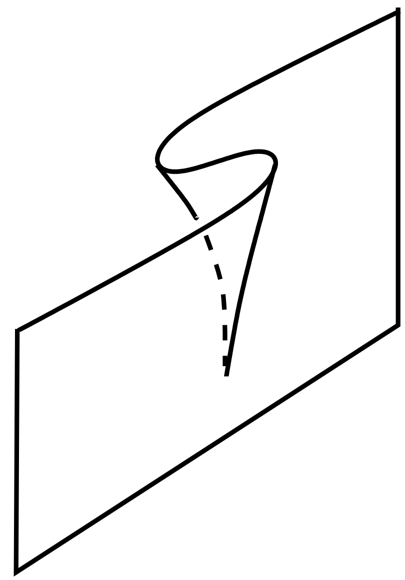
- 1d tangles in the 2-cube as morphisms:



- isotopy classes of 2d tangles in the 3-cube as 2-morphisms:



Objects in $\mathbf{2Tang_1}$ have duals — but now the zig-zag equations are weakened to 2-isomorphisms. These come from the *cusp catastrophe*:

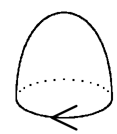


and they satisfy equations coming from the *swallowtail catastrophe*.

Everything **in nTang$_k$ has duals: not just objects, but j-morphisms for all j.**

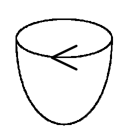
For example, in 2Tang$_1$ we have the unit of the counit of the point x:

$$i_{e_x}\colon 1_{1_I} \Rightarrow e_x e_x^*$$



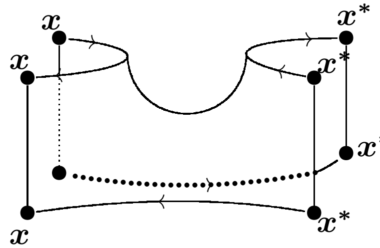
We also have the counit of the unit:

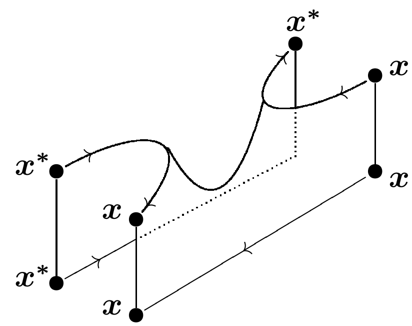
$$e_{i_x}\colon i_x^* i_x \Rightarrow 1_{1_I}$$



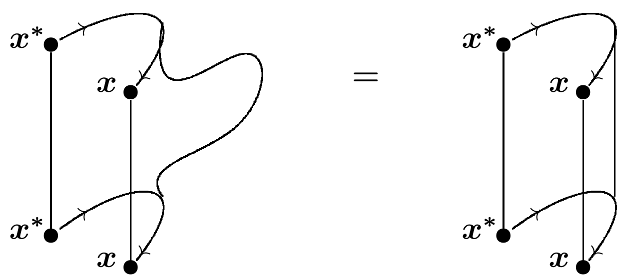
We also have

$$i_{i_x}\colon 1_{x\otimes x^*} \Rightarrow i_x i_x \qquad\qquad e_{e_x}\colon e_x^* e_x \Rightarrow 1_{x^*\otimes x}$$





All these give *critical points* in 2d Morse theory. The zig-zag equations then give *cancellation of critical points*, like this:



Even more interesting are 2d tangles in 4 dimensions. In 1997, Laurel Langford and I proved:

Theorem: $2\mathrm{Tang}_2$, the category of 2d tangles in a $(2+2)$-dimensional cube, is the free braided monoidal 2-category with duals on one object x.

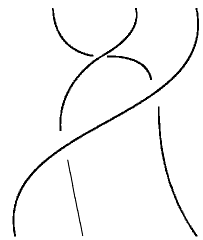
This categorifies ordinary knot theory! The category C of representations of a quantum group is a braided monoidal category with duals, so any object $a \in C$ gives a tangle invariant

$$Z\colon 1\mathrm{Tang}_2 \to C$$

with $Z(x) = a$.

If we categorify C and obtain a braided monoidal *2-category* with duals, say $\widetilde{C}$, any object $a \in \widetilde{C}$ gives a 2-tangle invariant $\widetilde{Z}\colon 2\mathrm{Tang}_2 \to \widetilde{C}$.

In a braided monoidal category the braiding satisfies the 'Yang–Baxter equation':



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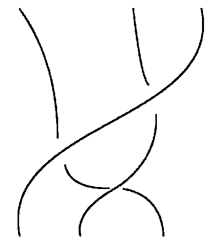
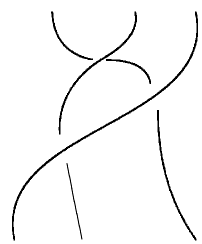
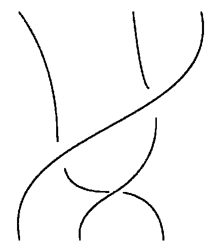


In a braided monoidal 2-category, this becomes a 2-isomorphism, the 'Yang–Baxterator':

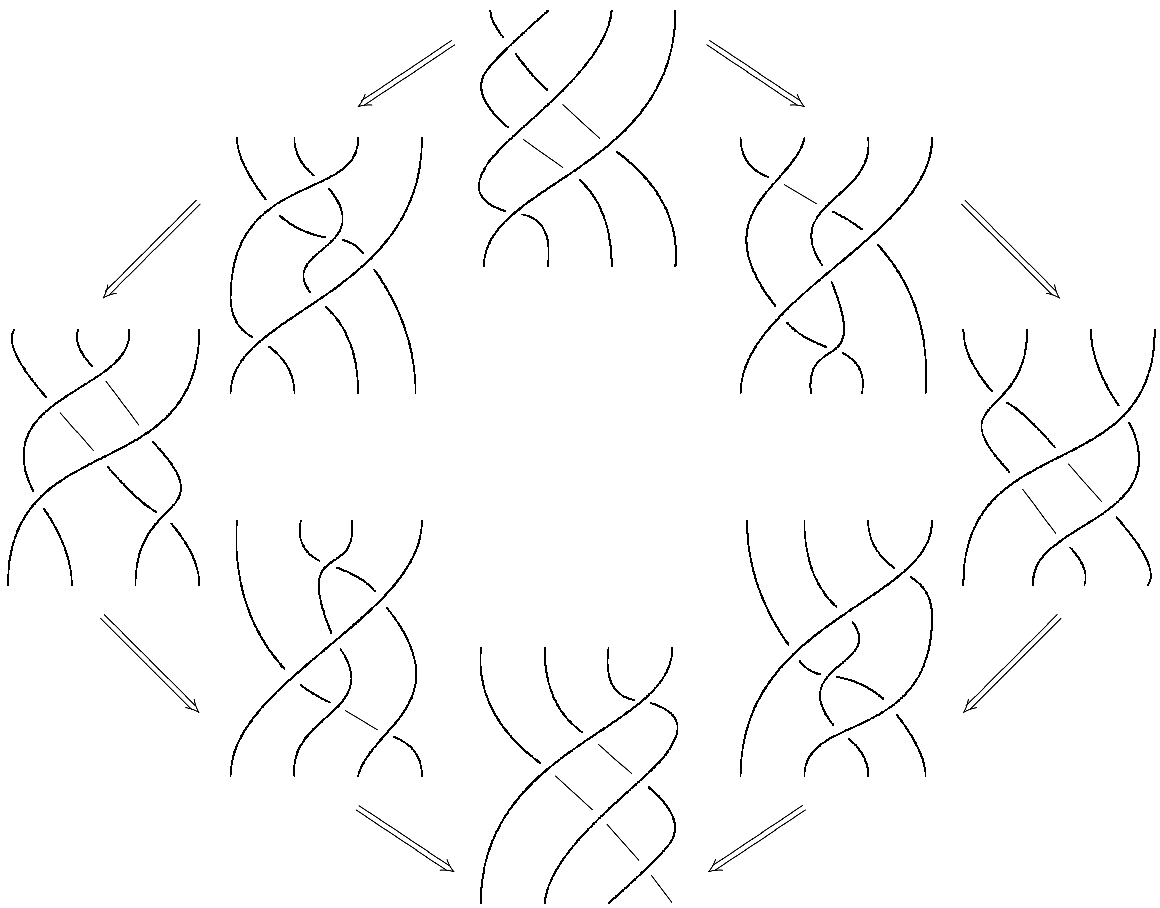
$\boldsymbol{Y}$ **:**



$\Rightarrow$



This in turn satisfies the 'Zamolodchikov tetrahedron equation':



More generally, in 1995 Dolan and I formulated:

The Tangle Hypothesis: $n\mathrm{Tang}_k$, the n-category of framed n-dimensional tangles in a $(k+n)$-dimensional cube, is the free k-tuply monoidal n-category with duals on one object x: the positively oriented point.

Taking the limit $k \to \infty$ and applying the Stabilization Hypothesis, this gives:

The Cobordism Hypothesis: $n\mathrm{Cob}$, the n-category of framed cobordisms, is the free stable n-category with duals on one object x.

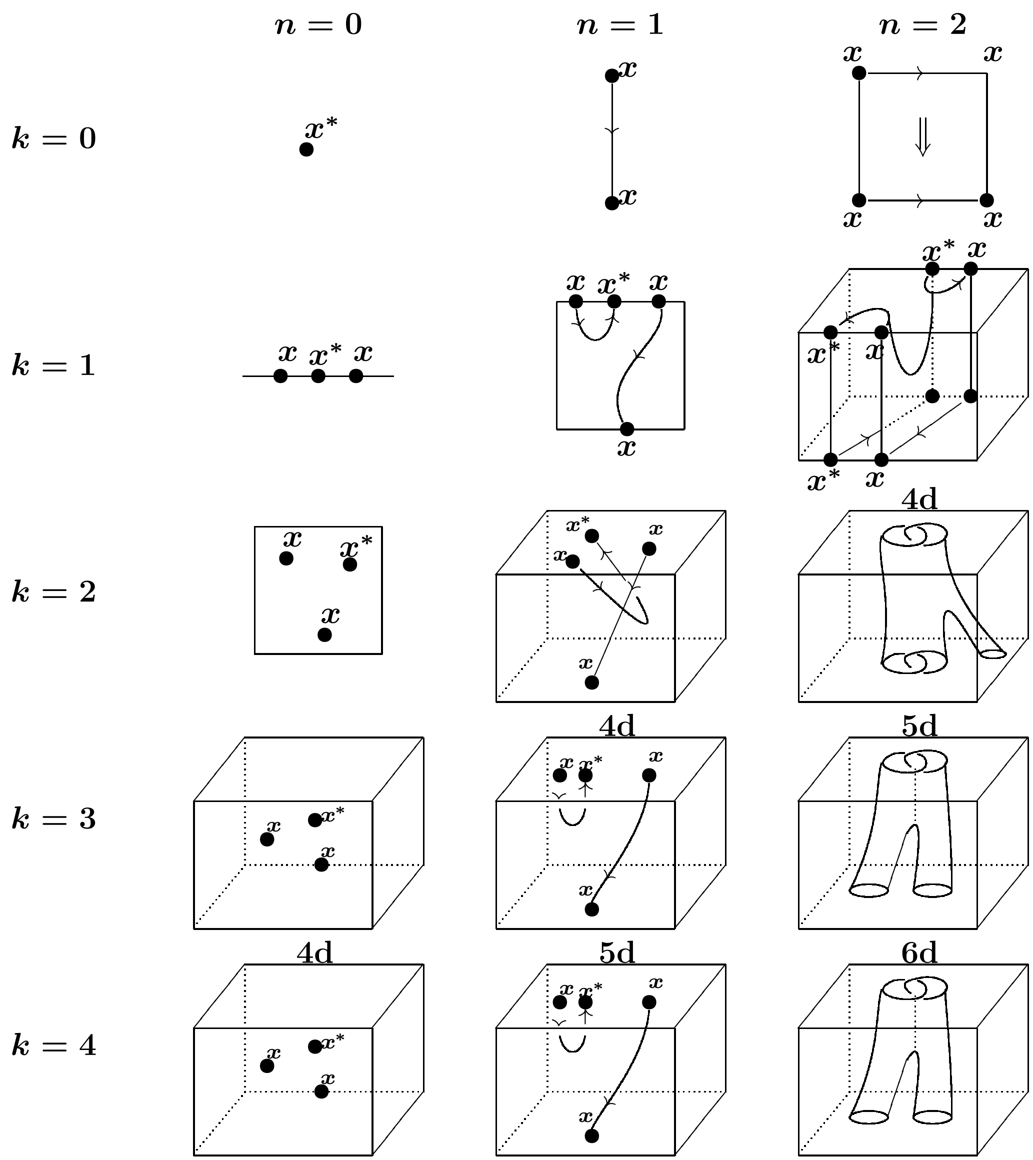
If the Cobordism Hypothesis is true, we can construct an 'n-dimensional extended TQFT', that is a stable n-functor

$$\boldsymbol{Z\colon n\mathrm{Cob} \to C,}$$

simply by choosing any stable n-category with duals C and any object $a \in C$.

Z is determined by its value on the point:

$$\boldsymbol{Z(x) = a.}$$



$n = 0$
$n = 1$
$n = 2$
$k = 0$
$k = 1$
$k = 2$
$k = 3$
$k = 4$
x^*
x
4d
5d
6d

Based on joint work with Mike Hopkins, Jacob Lurie announced a proof of the Cobordism Hypothesis in 2008. He reformulated it using (∞, n)-categories.

Roughly, an '(∞, n)-category' is an ∞-category where all j-morphisms with $j > n$ are invertible. nCob should be such a thing, with:

- **collections of oriented points as objects,**
- **framed 1d cobordisms between these as morphisms,**
- **framed 2d cobordisms between these as 2-morphisms,**
- **...**
- **framed nd cobordisms between these as n-morphisms,**
- **diffeomorphisms of these as $(n + 1)$-morphisms,**
- **smooth paths of diffeomorphisms as $(n+2)$-morphisms...**

A bit more formally:

An $(\infty, 0)$-category is just an ∞-groupoid. In the simplicial approach these are 'Kan complexes'. The Homotopy Hypothesis has been proved in this framework: the model category of Kan complexes is Quillen equivalent to the model category of topological spaces.

There is also a simplicial approach to $(\infty, 1)$-categories: 'complete Segal spaces'. There is a version of the Homotopy Hypothesis for these, too! The model category of complete Segal spaces is Quillen equivalent to the model category of 'topological categories': categories for which each set $\hom(x, y)$ is a topological space, and composition is continuous.

In 2005, Clark Barwick generalized complete Segal spaces to define (∞, n)-categories.

Lurie claims to prove:

The Cobordism Hypothesis: Let C be a stable (∞, n)-category. Then there is a bijection between equivalence classes of stable (∞, n)-functors

$$Z\colon n\mathrm{Cob} \to C$$

and equivalence classes of fully dualizable objects $a \in C$.

This bijection sends Z to

$$Z(x) = a$$

where $x \in n\mathrm{Cob}$ is the positively oriented point.

Lurie is also working on a version of the Tangle Hypothesis.

So: the challenge for algebraists is to *construct k-tuply monoidal n-categories with duals, and get invariants of manifolds and higher-dimensional knots!*

What's the real reason we can categorify quantum groups and their representations, and — apparently — get braided monoidal *2-categories* with duals? Can we get braided monoidal *3-categories*?

More generally: what naturally arising algebraic gadgets have representations forming k-tuply monoidal n-categories with duals? *What's the pattern?*

Two challenges for topologists: prove the Generalized Tangle Hypothesis, and construct the 'fundamental monoidal n-category with duals' of a stratified space.