Higher Gauge Theory and the String Group

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For more see: http://math.ucr.edu/home/baez/esi/
Categorification

- sets $\rightsquigarrow$ categories
- functions $\rightsquigarrow$ functors
- equations $\rightsquigarrow$ natural isomorphisms

Categorification ‘boosts the dimension’ by one:

In strict categorification we keep equations as equations. This is evil... but today we’ll do it whenever it doesn’t cause trouble, just to save time.
Higher Gauge Theory

groups $\rightsquigarrow$ 2-groups
Lie algebras $\rightsquigarrow$ Lie 2-algebras
bundles $\rightsquigarrow$ 2-bundles
connections $\rightsquigarrow$ 2-connections

Connections describe parallel transport for particles.
2-Connections describe parallel transport for strings!

![Diagram]

We should even go beyond $n = 2$... but not today.
Fix a simply-connected compact simple Lie group $G$. Then:

- The Lie algebra $\mathfrak{g}$ gives a 1-parameter family of Lie 2-algebras $\text{string}_k(\mathfrak{g})$.
- When $k \in \mathbb{Z}$, $\text{string}_k(\mathfrak{g})$ comes from a Lie 2-group $\text{String}_k(G)$.
- The ‘geometric realization of the nerve’ of $\text{String}_k(G)$ is a topological group, $|\text{String}_k(G)|$.
- Principal $\text{String}_k(G)$-2-bundles are the same as $|\text{String}_k(G)|$-bundles.
- For $k = 1$, $|\text{String}_k(G)|$ is $G$ with its 3rd homotopy group made trivial.
- We can define connections and characteristic classes for $\text{String}_k(G)$-2-bundles!
A strict 2-group is a category in Grp: a category with a group of objects and a group of morphisms, such that all the category operations are group homomorphisms.

We draw the objects in a 2-group like this:

```
• ——— •
g
```

We draw the morphisms like this:

```
• ——— •
  \   \  
  f   f
```

\[ \bullet \quad || \quad \bullet \]

\[ \bullet \quad \Rightarrow \quad \bullet \]

\[ g \quad \Rightarrow \quad g' \]
We can multiply objects: 

multiply morphisms: 

and compose morphisms:
All 3 operations have a unit and inverses. All 3 are associative, so these are well-defined:

Finally, the **interchange law** holds, meaning

is well-defined.
Lie 2-Algebras

A strict Lie 2-algebra is a category in LieAlg: a category with a Lie algebra of objects and a Lie algebra of morphisms, such that all the category operations are Lie algebra homomorphisms.

The theory of strict Lie 2-algebras closely mimics the theory of 2-groups. For example...
**Theorem** (Mac Lane, Sinh). A 2-group is determined up to equivalence by:

- the group $G$ of isomorphism classes of objects,
- the abelian group $A$ of endomorphisms of any object,
- an action of $G$ on $A$,
- an element of $H^3(G, A)$.

**Theorem** (Gerstenhaber, Crans). A Lie 2-algebra is determined up to equivalence by:

- the Lie algebra $\mathfrak{g}$ of isomorphism classes of objects,
- the vector space $\mathfrak{a}$ of endomorphisms of any object,
- a representation of $\mathfrak{g}$ on $\mathfrak{a}$,
- an element of $H^3(\mathfrak{g}, \mathfrak{a})$. 
Suppose $G$ is a simply-connected compact simple Lie group. Let $\mathfrak{g}$ be its Lie algebra. A lemma of Whitehead says:

$$H^3(\mathfrak{g}, \mathbb{R}) = \mathbb{R}$$

So:

**Corollary.** For any $k \in \mathbb{R}$ there is a Lie 2-algebra $\text{string}_k(\mathfrak{g})$ for which:

- $\mathfrak{g}$ is the Lie algebra of isomorphism classes of objects;
- $\mathbb{R}$ is the vector space of endomorphisms of any object.

Every Lie 2-algebra with these properties is equivalent to $\text{string}_k(\mathfrak{g})$ for some unique $k \in \mathbb{R}$. 
**Theorem.** For any \( k \in \mathbb{Z} \), \( \text{string}_k(g) \) is the Lie 2-algebra of an infinite-dimensional Lie 2-group \( \text{String}_k(G) \).

**Theorem.** The morphisms in \( \text{String}_k(G) \) starting at the constant path form the level-\( k \) central extension of the loop group \( \Omega G \):

\[
1 \longrightarrow U(1) \longrightarrow \widehat{\Omega_k G} \longrightarrow \Omega G \longrightarrow 1
\]
For any category $\mathcal{C}$ there is a space $|\mathcal{C}|$, the **geometric realization of the nerve** of $\mathcal{C}$, built from a vertex for each object:

- $\bullet x$

an edge for each morphism:

- $\bullet \xrightarrow{f} \bullet$

a triangle for each composable pair of morphisms:

```
  \bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet
```

$fg$

a tetrahedron for each composable triple:

```
  \bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet \xrightarrow{h} \bullet
```

$fg$, $gh$, $fgh$

and so on...
A 2-group is a category with a product and inverses. So, if $\mathcal{G}$ is a 2-group, $|\mathcal{G}|$ is a topological group.

More generally, we can define a topological group $|\mathcal{G}|$ for any topological 2-group $\mathcal{G}$.

**Theorem.** For any $k \in \mathbb{Z}$, there is a short exact sequence of topological groups:

$$1 \longrightarrow K(\mathbb{Z}, 2) \longrightarrow |\text{String}_k(G)| \xrightarrow{p} G \longrightarrow 1$$

where $p$ is a fibration. When $k = 1$, this exhibits $|\text{String}_k(G)|$ as the ‘3-connected cover’ of $G$: the topological group formed by making the 3rd homotopy group of $G$ trivial.
For example, start with $O(n)$:

- Making $\pi_0$ trivial gives $SO(n)$.
- Making $\pi_1$ trivial gives $\text{Spin}(n)$.
- Making $\pi_2$ trivial still gives $\text{Spin}(n)$.
- Making $\pi_3$ trivial gives the 3-connected cover... something new and interesting: the \textbf{string group}.

So, we’re getting the string group from a 2-group.
2-Bundles — Quick and Dirty

For any topological 2-group $\mathcal{G}$ and any space $X$, we can define a principal $\mathcal{G}$-2-bundle over $X$ to consist of:

- an open cover $U_i$ of $X$,
- continuous maps $g_{ij}: U_i \cap U_j \to \text{Ob}(\mathcal{G})$ satisfying $g_{ii} = 1$, and
- continuous maps $h_{ijk}: U_i \cap U_j \cap U_k \to \text{Mor}(\mathcal{G})$ with

$$h_{ijk}(x): g_{ij}(x)g_{jk}(x) \to g_{ik}(x)$$
satisfying the **nonabelian 2-cocycle condition**:

\[
\begin{array}{c}
\begin{array}{c}
g_{ij} \\
g_{ik} \\
g_{il}
\end{array}
\rightarrow
\begin{array}{c}
g_{jk} \\
h_{ijk} \\
g_{kl}
\end{array}
\rightarrow
\begin{array}{c}
g_{ik} \\
h_{ikl} \\
g_{il}
\end{array}
\end{array}
= 
\begin{array}{c}
\begin{array}{c}
g_{ij} \\
g_{jl} \\
g_{il}
\end{array}
\rightarrow
\begin{array}{c}
g_{jk} \\
h_{jkl} \\
g_{kl}
\end{array}
\rightarrow
\begin{array}{c}
g_{jl} \\
h_{ijl} \\
g_{il}
\end{array}
\end{array}
\]

on any quadruple intersection \( U_i \cap U_j \cap U_k \cap U_\ell \).
There’s a natural notion of ‘equivalence’ for 2-bundles over $X$, since they form a 2-category.

**Theorem.** For any topological 2-group $G$ and paracompact Hausdorff space $X$, there is a 1-1 correspondence between:

- equivalence classes of principal $G$-2-bundles over $X$,
- isomorphism classes of principal $|G|$-bundles over $X$,
- homotopy classes of maps $f : X \to B|G|$.

So, $B|G|$ is the classifying space for $G$-2-bundles.
Characteristic Classes

Let $G$ be a simply-connected compact simple Lie group, and let $\mathcal{G} = \text{String}_k(G)$ with $k = 1$.

The homomorphism

$$|\mathcal{G}| \xrightarrow{p} G$$

gives an algebra homomorphism:

$$H^*(BG, \mathbb{R}) \xrightarrow{p^*} H^*(B|\mathcal{G}|, \mathbb{R})$$

This is onto, with kernel generated by the ‘second Chern class’ $c_2 \in H^4(BG, \mathbb{R})$.

In this case, the real characteristic classes of $\mathcal{G}$-2-bundles are just like those of $G$-bundles, but with the second Chern class killed!
There’s a concept of ‘connection’ and ‘curvature’ for 2-bundles when $\mathcal{G}$ is a Lie 2-group.

In this case, we should be able to compute the real characteristic classes of a 2-bundle starting from a connection on this 2-bundle.

Sati, Schreiber, and Stasheff have already made progress towards this.