A Compositional Framework for Passive Linear Networks

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5th October 2014

Abstract

This paper centres around two dagger compact categories and the relationship between them. The first is a category whose morphisms are electrical circuits comprising resistors, inductors, and capacitors, with marked input and output terminals. To understand this category, we begin by recalling Ohm’s law and Kirchhoff’s laws, and show that we may also understand these circuits through a formal minimisation principle, a generalisation of the so-called principle of minimum power. This suggests an equivalence relation on circuits by ‘black boxing’—that is, treating circuits as some sort of ‘black box’ whose terminals we can access, but whose inner workings we cannot. Mathematically, this involves representing circuits by their power consumption, described by a type of quadratic form known as a Dirichlet form. While a composition law can be defined for these forms, we see that more work is required for identity morphisms to exist, leading us to the second category central to our discussion: the category of symplectic vector spaces and Lagrangian relations.

The end goal is then to prove the existence of a functor, dubbed the ‘black box functor’, from the category of circuits to the category of Lagrangian relations. To assist with the proof, we turn to the concepts of cospans and corelations, developing the technical machinery required to turn a lax monoidal functor with source the category of finite sets with monoidal product disjoint union into a dagger compact category of decorated cospans.

We write for a reader with basic familiarity with monoidal categories, but assume none with circuit theory or symplectic linear algebra.

This is a draft article. Please send any feedback to Brendan Fong at brendan.fong@cs.ox.ac.uk. Requests for clarification, corrections, and suggestions for improvement are very much appreciated.

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1 Introduction

The 1950s saw Feynman develop his diagrams in for processes in physics, and Eilenberg and Mac
Lane develop the beginnings of category theory. Over the subsequent decades, culminating in the
work of Joyal and Street in the 1980s, it became clear that these developments were profoundly
linked: monoidal categories have precise graphical representation through string diagrams, and con-
versely monoidal categories provide an algebraic foundation for the geometric intuition of Feynman
diagrams. This relationship has given and continues to give insight into physics and mathematics
alike; more detail can be found in Baez and Lauda’s prehistory of \( n \)-categorical physics [5].

For us, here, the key emergent insight is the use of category theory in applications where mor-
phisms are thought of explicitly as physical processes, rather than structure-preserving maps between
algebraic objects. Since then, this approach has filtered into more immediate applications, particular-
inly in computation and quantum computation [1, 3, 48]. This paper proposes to contribute to
a still nascent program of work [53, 51, 4] pursuing these ideas in network engineering and science,
with the aim of giving diverse network-type diagrams compatible foundations based on monoidal
categories.

Indeed, just as physicists were using Feynman diagrams, albeit without reference to their formali-
sation as monoidal categories, branches of engineering were using many types of diagrams them-
selves. The foremost among these is the ubiquitous concept of an electrical circuit diagram. Although often
less well-known, similar diagrams are used in many contexts in which notions of effort and flow are
present in network-like structures, such as in hydraulic systems or electrical circuits, and certain
mechanical constructions. First discussed in detail by Olsen [38], mathematically precise analogies
exist between, for example, systems of the types given by the following table, and their associated
quantities.

<table>
<thead>
<tr>
<th></th>
<th>displacement</th>
<th>flow</th>
<th>momentum</th>
<th>effort</th>
</tr>
</thead>
<tbody>
<tr>
<td>Electronics</td>
<td>charge</td>
<td>( q )</td>
<td>flux linkage</td>
<td>voltage</td>
</tr>
<tr>
<td>Translation</td>
<td>position</td>
<td>( q' )</td>
<td>velocity</td>
<td>momentum</td>
</tr>
<tr>
<td>Rotation</td>
<td>angle</td>
<td>( p )</td>
<td>angular velocity</td>
<td>force</td>
</tr>
<tr>
<td>Hydraulics</td>
<td>volume</td>
<td>( p' )</td>
<td>angular moment</td>
<td>torque</td>
</tr>
<tr>
<td>Thermodynamics</td>
<td>entropy</td>
<td>flow</td>
<td>temperature momentum</td>
<td>temperature</td>
</tr>
<tr>
<td>Chemistry</td>
<td>moles</td>
<td>molar flow</td>
<td>chemical momentum</td>
<td>chemical potential</td>
</tr>
</tbody>
</table>

Further work, pioneered in particular by schools led by Odum and Forrester, links these network-
type systems to biology, ecology, and economics [37, 21]. We do not, however, at first expect this will
help electrical engineers and other users of these diagrams—the engineering of all that we discuss in
this paper is more than well-known—but we hope that these new perspectives will give fresh light
and new examples for aspects of category theory.

Although we keep this broad applicability in the back of our minds, we couch the present discus-
sion in terms of electrical circuits for reasons of familiarity, and start at the most basic of components:
resistors. Our objects of study then look like this:

![Resistor Diagram](image)

To formalise this, define an open circuit of linear resistors to comprise a labelled graph \((N, E, s, t, r)\)
and a pair of functions \(i : X \rightarrow N\), \(o : Y \rightarrow N\), where \(N, E, X, Y\) are finite sets, and \(s, t : E \rightarrow N, r : E \rightarrow (0, \infty)\) are functions. We think of \(N\) as the set of nodes of the circuit, \(E\) as the set of
resistors, \(s, t\) as declaring the source and target node of each resistor, \(r\) as declaring the resistance,
and $X$ and $Y$ as the set of inputs and outputs respectively. Then, as we shall see in detail, the possible electrical states of the circuit are in one-to-one correspondence with elements of the vector space $\mathbb{R}^{\partial N}$, which assign a real number, thought of as an electric potential, to each element in the set $\partial N = i(X) \cup o(Y)$, thought of as a terminal of the circuit. This correspondence is specified by the so-called power functional or Dirichlet form $Q: \mathbb{R}^{\partial N} \rightarrow \mathbb{R}$ associated to the circuit

$$Q(\psi) = \min_{\phi \in \mathbb{R}^N} \sum_{e \in E} \frac{1}{r(e)} (\phi(t(e)) - \phi(s(e)))^2;$$

the current corresponding to a given potential is the one that minimizes the above function—a statement of the so-called principle of minimum power.

The Laplace transform allows us to generalise this immediately to the setting of passive linear circuits—that is, the case in which we allow circuits to contain inductors and capacitors too—simply by changing the field we work over from $\mathbb{R}$ to the field $\mathbb{R}(s)$ of real rational functions of a single variable, and talking of impedence where we previously talked of resistance. We call the resulting category of circuit diagrams $\text{Circ}$, where objects are finite sets, a morphism $X \rightarrow Y$ is a circuit with input set $X$ and output set $Y$, and composition is given by identifying the outputs of one circuit with the inputs of the next, and taking the resulting union of the labelled graphs. The above Dirichlet forms classify circuits up to equivalent electrical behaviour on the terminals.

We wish for these equivalence classes of circuits to form a category. Although there is a notion of composition for Dirichlet forms, we find that it lacks identity morphisms or, equivalently, it lacks morphisms representing ideal wires of zero impedence. To address this we turn to Lagrangian subspaces of symplectic vector spaces, a generalisation of quadratic forms via the map $(Q: F^{\partial N} \rightarrow F) \mapsto \text{Graph}(dQ) = \{(\psi, dQ_\psi) \mid \psi \in F^{\partial N}\} \subseteq F^{\partial N} \oplus (F^{\partial N})^*$ taking a quadratic form $Q$ on a set $\partial N$ over a field $F$ to the graph of its differential $dQ$. Here we think of the symplectic vector space $F^{\partial N} \oplus (F^{\partial N})^*$ as the state space of the circuit, and the subspace $\text{Graph}(dQ)$ as the subspace of attainable states, with $\psi \in F^{\partial N}$ detailing the potentials at the terminals, and $dQ_\psi \in (F^{\partial N})^*$ the currents. This construction is well-known in classical mechanics, where the principle of least action plays a role analogous to that of the principle of minimum power here, and the sets of quadratic forms and Lagrangian subspaces have natural topologies such that the set of Lagrangian subspaces contains all limits of sequences of quadratic forms. In particular, this means that we may find identity morphisms for our composition of Dirichlet forms by sending impedances to zero. Moreover, there exists a category $\text{LagrRel}$ with objects finite dimensional symplectic vector spaces and morphisms Lagrangian relations: relations $V \rightarrow W$ that are given by Lagrangian subspaces of $V \oplus W$, where $V$ is the symplectic vector space conjugate to $V$.

To move from the Lagrangian subspace defined by the graph of the differential of the power functional to a morphism in the category $\text{LagrRel}$—that is, to a Lagrangian relation—, we must treat seriously the input and output functions of the circuit. These realise the circuit as built upon a cospan

$$N \xymatrix{ i \ar[r] \ar[dr] & X \ar[dl] \ar[r] & Y \ar[l] \ar[ddr] \ar[dl] & o \ar[l] }$$

Applicable far more broadly than this present formalisation of circuits, cospans model systems with two ‘ends’, an input and output end, albeit without any connotation of directionality: we might just as well exchange the role of the inputs and outputs by taking the mirror image of the above diagram. The role of the input and output functions, as we have discussed, is to mark the terminals we may glue onto the terminals of another circuit, and the pushout of cospans gives formal precision to this gluing construction.
One upshot of this cospan framework is that we may consider circuits with elements of \( N \) that are both inputs and outputs, such as this one:

![Diagram of a circuit with a single 2Ω resistor](image)

This corresponds to the identity morphism on the finite set with two elements. Another is that some points may be considered an input or output multiple times; we draw this:

![Diagram of a circuit with multiple inputs and outputs](image)

This allows us to connect two distinct output terminals to the above double input terminal.

As is the case for the nodes of a circuit, given a set \( X \) of inputs or outputs, we understand the electrical behaviour on these sets by considering the symplectic vector space \( \mathbb{F}^X \oplus (\mathbb{F}^X)^* \), arising from the direct sum of the space \( \mathbb{F}^X \) of potentials and the space \((\mathbb{F}^X)^* \) of currents at these points. A Lagrangian relation specifies which states of the output space \( \mathbb{F}^Y \oplus (\mathbb{F}^Y)^* \) are possible behaviours compatible with each state of the input space \( \mathbb{F}^X \oplus (\mathbb{F}^X)^* \). Turning the Lagrangian subspace \( \text{Graph}(dQ) \) of a circuit into this information requires that we understand the symplectifications \( Sf : \mathbb{F}^B \oplus (\mathbb{F}^B)^* \rightarrow \mathbb{F}^A \oplus (\mathbb{F}^A)^* \) and \( S^t f : \mathbb{F}^B \oplus (\mathbb{F}^B)^* \rightarrow \mathbb{F}^A \oplus (\mathbb{F}^A)^* \) of functions \( f : A \rightarrow B \). In particular we need to understand the symplectifications of our input and output functions with codomain restricted to \( \partial N \); abusing notation, we also write these \( i : X \rightarrow \partial N \) and \( o : Y \rightarrow \partial N \).

These symplectifications are themselves Lagrangian relations, and the catch phrase is that they ‘copy voltages’ and ‘split currents’. More precisely, for any given potential-current pair \((\psi, \iota)\) in \( \mathbb{F}^B \oplus (\mathbb{F}^B)^* \), its image under \( Sf \) comprises all elements of \((\psi', \iota') \in \mathbb{F}^A \oplus (\mathbb{F}^A)^*\) such that the potential at \( a \in A \) is equal to the potential at \( f(a) \in B \), and such that, for each fixed \( b \in B \), collectively the currents at the \( a \in f^{-1}(b) \) sum to the current at \( b \). We use the symplectification \( So \) of the output function to transform the behaviour on \( \partial N \) to that on the outputs \( Y \). As our current framework is set up to report the flow of charge out of each node, to describe input currents we define the twisted symplectification \( S^t f : \mathbb{F}^B \oplus (\mathbb{F}^B)^* \rightarrow \mathbb{F}^A \oplus (\mathbb{F}^A)^* \) almost identically to the above, but for the exception that we flip the sign of the currents \( \iota' \in (\mathbb{F}^A)^* \), and use the twisted symplectification \( S^t i \) of the input function.

The Lagrangian relation corresponding to a circuit then comprises exactly a list of the potential–current pairs that are possible electrical states of the inputs and outputs of the circuit. In doing so, it identifies distinct circuits. A simple example of this is the identification of a single 2Ω resistor

![Diagram of a circuit](image)

with two 1Ω resistors in series

![Diagram of a circuit](image)

The inability to access the internal workings of a circuit in this representation inspires us to call this process ‘black boxing’, evoking the image of encasing circuitry in an opaque black box, leaving only the terminals accessible. Fortunately, this information is enough to completely characterise the electrical behaviour of a circuit, including how it interacts when connected with our circuits.

Put more precisely, the black boxing process is functorial:
Theorem. There exists a strong monoidal dagger functor $\Box : \text{Circ} \to \text{LagrRel}$ mapping a finite set $X$ to the symplectic vector space $\mathbb{F}^X \oplus (\mathbb{F}^X)^*$ it generates, and a circuit $((N,E,s,t,r),i,o)$ to the Lagrangian relation

$$\bigcup_{v \in \text{Graph}(dQ)} S^t i(v) \times S o(v) \subseteq \mathbb{F}^X \oplus (\mathbb{F}^X)^* \oplus \mathbb{F}^Y \oplus (\mathbb{F}^Y)^*, $$

where $Q$ is the function assigning to each boundary voltage the power emitted by the circuit.

The goal of this paper is to prove this.

1.1 Finding your way through this paper

This paper is split into three parts, addressing in turn the questions:

I. What do circuit diagrams mean?

II. How do we interact with circuit diagrams?

III. How is meaning preserved under these interactions?

We begin Part I, on the semantics of circuit diagrams, with a discussion of circuits of linear resistors, developing the intuition for the governing laws of passive linear circuits—Ohm’s law, Kirchhoff’s voltage law, and Kirchhoff’s current law—in a time-independent setting (Section 2). This in particular allows us to develop the concept of Dirichlet form as a representation of power consumption, and understand their composition as minimising power, an expression of the current law. In Section 3, the Laplace transform then allows us to recapitulate these ideas after introducing inductors and capacitors, talking of impedance where we formerly talked of resistance, and generalising Dirichlet forms from the field $\mathbb{R}$ to the field $\mathbb{R}(s)$ of real rational functions. While in this setting the interpretations these concepts have in the realm of resistors no longer holds, the intuitions gained there still remain useful. Nonetheless, we find that Dirichlet forms alone do not provide the flexibility to construct a category representing the semantics of circuit diagrams. This motivates the development of more powerful machinery.

Part II, on the syntax of circuit diagrams, contains the main technical contributions of the paper. It begins with Section 4, which develops machinery to construct what we call categories of decorated cospans. These are categories where the objects are finite sets and the morphisms are cospans in the category of finite sets together with some extra structure on the apex. Circuits, as defined above, are naturally an example of such a construction, and Section 5 then lays out the details of this. In Section 6 we then review the basic theory of linear Lagrangian relations, giving details to the correspondence we have defined between Dirichlet forms, and hence passive linear circuits, and Lagrangian relations. Section 7 then takes immediate advantage of the added flexibility of Lagrangian relations, discussing the ‘trivial’ circuits comprising only wires of zero impedance, which mediate the notion of composition of circuits.

Having developed the prerequisites, Part III then gets straight to the point, reminding us of the definition and interpretation of the black box functor (Section 8), and then proving its functoriality (Section 9).

Acknowledgements

BF would like to thank the Clarendon Fund, University of Oxford, the Centre for Quantum Technologies, Singapore, and the first author for their support.
Part I
Passive Linear Circuits

In this part we review the meaning of circuit diagrams comprising resistors, inductors, and capacitors, giving an answer to the question “What do circuit diagrams mean?”. To elaborate, while circuit diagrams model electric circuits according to their physical form, another, often more relevant, way to understand a circuit is by its electrical behaviour. This means the following. To an electric circuit we associate two quantities to each edge: voltage and current. We are not free, however, to choose these quantities as we like; circuits are subject to governing laws that imply voltages and currents must obey certain relationships. From the perspective of control theory we are particularly interested in the values these quantities take at the so-called terminals, and how altering one value will affect the other values. We call two circuits equivalent when they determine the same relationship. Our main task in this first part is to explore when two circuits are equivalent.

2 Circuits of linear resistors

In order to let physical intuition lead the way, we begin by specialising to the case of linear resistors. In this section we describe how to find the function of a circuit from its form, advocating in particular the perspective of the principle of minimum power. This allows us to identify the behaviour of a circuit with a so-called Dirichlet form representing the dependence of its power consumption on potentials at its terminals.

2.1 Circuits as labelled graphs

The concept of an abstract open electrical circuit made of linear resistors is well-known in electrical engineering, but we shall need to formalize it with more precision than usual. The basic idea is that a circuit of linear resistors is a graph whose edges are labelled by positive real numbers called ‘resistances’, and whose sets of vertices is equipped with two subsets: the ‘inputs’ and ‘outputs’. This unfolds as follows.

A (closed) circuit of resistors looks like this:

![Closed circuit diagram](image)

We can consider this a labelled graph, with each resistor an edge of the graph, its resistance its label, and the vertices of the graph the points at which resistors are connected.

A circuit is open if it allows connections with other circuits. To do this we first mark points at which connections can be made by denoting some vertices as input and output terminals:

![Open circuit diagram](image)
Then, given a second circuit, we may choose a relation between the output set of the first and the input set of this second circuit, such as the simple relation of the single output vertex of the circuit above with the single input vertex of the circuit below.

We connect the two circuits by identifying output and input vertices according to this relation, giving in this case the composite circuit:

More formally, define a graph\(^1\) to be a pair of functions \(s, t : E \to N\) where \(E\) and \(N\) are finite sets. Given a finite set \(L\) of labels, a \(L\)-graph is a graph further equipped with a function \(r : E \to L\).

We call elements of \(E\) edges and elements of \(N\) vertices or nodes. We say that the edge \(e \in E\) has source \(s(e)\) and target \(t(e)\), and also say that \(e\) is an edge from \(s(e)\) to \(t(e)\).

We may then make a definition of circuit.

**Definition 2.1.** Given a set \(L\), a circuit over \(L\) \((\Gamma, i, o)\) is an \(L\)-graph \(\Gamma = (E, N, s, t, r)\), together with finite sets \(X\), \(Y\), and functions \(i : X \to N\) and \(o : Y \to N\). We call the sets \(i(X)\), \(o(Y)\), and \(\partial N = i(X) \cup o(Y)\) the inputs, outputs, and terminals or boundary of the circuit respectively.

We will later make use of the notion of connectedness in graphs. Recall that given two vertices \(v, w \in N\) of a graph, a path from \(v\) to \(w\) is a finite sequence of vertices \(v = v_0, v_1, \ldots, v_n = w\) and edges \(e_1, \ldots, e_n\) such that for each \(1 \leq i \leq n\), either \(e_i\) is an edge from \(v_i\) to \(v_{i+1}\), or an edge from \(v_{i+1}\) to \(v_i\). A subset \(S\) of the vertices of a graph is connected if, for each pair of vertices in \(S\), there is a path from one to the other. A connected component of a graph is a maximal connected subset of its vertices.\(^2\)

**Notation 2.2.** In the following let the \(L\)-labelled graph \(\Gamma = (E, N, s, t, r)\), together with input and output maps \(i : X \to N\) and \(o : Y \to N\), where \(X\) and \(Y\) are finite sets, denote a circuit. In particular \(E\) will always stand for the edge set of the apex of a circuit, \(N\) for the set of nodes, and \(\partial(N) = i(X) \cup o(Y)\) the set of terminals. For a finite set \(S\) and field \(\mathbb{F}\), we shall write \(\mathbb{F}^S\) for the vector space of functions \(\psi : S \to \mathbb{F}\). Elementary circuit theory is most frequently presented in terms of voltages and currents on the edges or ‘wires’ of the circuit. These are functions \(V \in \mathbb{R}^E\) and \(I \in \mathbb{R}^E\) respectively.\(^3\)

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\(^1\)Note that in this paper we refer to directed multigraphs simply as graphs.

\(^2\)Note that in the theory of directed graphs the qualifier ‘strongly’ is commonly used before the word ‘connected’ in these two definitions. As we never consider any other sort of connectedness, we omit this qualifier.

\(^3\)We use \(I\) for ‘intensity of current’, following Ampère.
2.2 Ohm’s law, Kirchhoff’s laws, and the principle of minimum power

In 1827 Georg Ohm published a book which included a relation between the voltage and current for circuits made of resistors [36]. At the time, the critical reception was harsh: one contemporary called Ohm’s work “a web of naked fancies, which can never find the semblance of support from even the most superficial of observations”, and the German Minister of Education said that a professor who preached such heresies was unworthy to teach science [16, 23]. However, a simplified version of his relation is now widely used under the name of “Ohm’s law”. We say that **Ohm’s law** holds if for all edges \( e \in E \) the voltage and current functions of a circuit obey:

\[
V(e) = r(e)I(e).
\]

Kirchhoff’s laws date to Gustav Kirchhoff in 1845, generalising Ohm’s work. They were in turn generalised into Maxwell’s equations a few decades later. We say **Kirchhoff’s voltage law** holds if there exists \( \phi \in \mathbb{R}^N \) such that

\[
V(e) = \phi(t(e)) - \phi(s(e)).
\]

We call the function \( \phi \) a **potential**, and think of it as assigning an electrical potential to each node in the circuit. The voltage then arises as the differences in potentials between adjacent nodes. If Kirchhoff’s voltage law holds for some voltage \( V \), the potential \( \phi \) is unique only in the trivial case of the empty circuit: when the set of nodes \( N \) is empty. Indeed, two potentials define the same voltage function if and only if their difference is constant on each connected component of the graph \( \Gamma \).

We say **Kirchhoff’s current law** holds if for all nonterminal nodes \( n \in N \setminus \partial N \) we have

\[
\sum_{s(e)=n} I(e) = \sum_{t(e)=n} I(e).
\]

This is an expression of conservation of charge within the circuit; it says that the total current flowing in or out of any nonterminal node is zero. Even when Kirchhoff’s current law is obeyed, terminals need not be sites of zero net current; we call the function \( \iota \in \mathbb{R}^{\partial N} \) that takes a terminal to the difference between the outward and inward flowing currents,

\[
\iota : \partial N \rightarrow \mathbb{R}
\]

\[
n \mapsto \sum_{t(e)=n} I(e) - \sum_{s(e)=n} I(e),
\]

the **boundary current** for \( I \).

A **boundary potential** is also a function in \( \mathbb{R}^{\partial N} \), but instead thought of as specifying electric potentials on the terminals of a circuit. As we think of our circuits as open circuits, with the terminals points of interaction with the external world, we shall think of these potentials as variables that are free for us to choose. Using the above three principles—Ohm’s law, Kirchhoff’s voltage law, and Kirchhoff’s current law—it is possible to show that choosing a boundary potential determines unique voltage and current functions on that circuit.

The so-called principle of minimum power gives some insight into how this occurs, by describing a way potentials on the terminals might determine potentials at all nodes. From this, Kirchhoff’s voltage law then gives rise to a voltage function on the edges, and Ohm’s law gives us a current function too. We shall show, in fact, that a potential satisfies the principle of minimum power for a given boundary potential if and only if this current obeys Kirchhoff’s current law.

A circuit with current \( I \) and voltage \( V \) dissipates energy at a rate proportional to the real number

\[
P = \sum_{e \in E} I(e)V(e).
\]

Ohm’s law allows us to rewrite \( I \) as \( V/r \), while Kirchhoff’s voltage law gives us a potential \( \phi \) such that \( V(e) \) can be written as \( \phi(t(e)) - \phi(s(e)) \), so for a circuit obeying these two laws the power can
also be expressed in terms of this potential. We thus arrive at a functional mapping potentials $\phi$ to the power dissipated by the circuit when Ohm’s law and Kirchhoff’s voltage law are obeyed for $\phi$.

**Definition 2.3.** The extended power functional $P : \mathbb{R}^N \to \mathbb{R}$ of a circuit is the map

$$P(\varphi) = \sum_{e \in E} \frac{1}{r(e)} \left( \varphi(t(e)) - \varphi(s(e)) \right)^2.$$ 

We call this the extended power functional as we shall see that it is defined on potentials that are not compatible with the three governing laws of electric circuits. We shall later restrict the domain of this functional so that it is defined precisely on those potentials that are compatible with the governing laws. Note that this functional does not depend on the directions chosen for the edges of the circuit.

This expression lets us formulate the ‘principle of minimum power’, which gives us information about the potential $\varphi$ given its restriction to the boundary of $\Gamma$. Call a potential $\varphi \in \mathbb{R}^\partial \mathbb{N}$ an extension of a boundary potential $\psi \in \mathbb{R}^\partial \mathbb{N}$ if $\varphi$ is equal to $\psi$ when restricted to $\mathbb{R}^\partial \mathbb{N}$—that is, if $\varphi|_{\partial \mathbb{N}} = \psi$.

**Definition 2.4.** We say a potential $\varphi \in \mathbb{R}^N$ obeys the principle of minimum power for a boundary potential $\psi \in \mathbb{R}^\partial \mathbb{N}$ if $\varphi$ minimizes the extended power functional $P$ subject to the constraint that $\varphi$ is an extension of $\psi$.

As promised, in the presence of Ohm’s law and Kirchhoff’s voltage law, the principle of minimum power is equivalent to Kirchhoff’s current law.

**Proposition 2.5.** Let $\varphi$ be a potential extending some boundary potential $\psi$. Then $\varphi$ obeys the principle of minimum power for $\psi$ if and only if the induced current $I(e) = \frac{1}{r(e)}(\varphi(t(e)) - \varphi(s(e)))$ obeys Kirchhoff’s current law.

**Proof.** Fixing the potentials at the terminals to be those given by the boundary potential $\psi$, the power is a nonnegative quadratic function of the potentials at the nonterminals. This implies that an extension $\varphi$ of $\psi$ minimises $P$ precisely when $\frac{\partial P}{\partial \varphi(n)}|_{\varphi=\psi} = 0$ for all nonterminals $n \in N \setminus \partial N$. Note that the partial derivative of the power with respect to the potential at $n$ is given by

$$\frac{\partial P}{\partial \varphi(n)}|_{\varphi=\psi} = \sum_{t(e)=n} \frac{1}{r(e)} (\varphi(t(e)) - \varphi(s(e))) - \sum_{s(e)=n} \frac{1}{r(e)} (\varphi(t(e)) - \varphi(s(e)))$$

$$= 2 \left( \sum_{t(e)=n} I(e) - \sum_{s(e)=n} I(e) \right).$$

Thus $\varphi$ obeys the principle of minimum power for $\psi$ if and only if $\sum_{s(e)=n} I(e) = \sum_{t(e)=n} I(e)$ for all $n \in N \setminus \partial N$, and so if and only if Kirchhoff’s current law holds. \qed

### 2.3 A Dirichlet problem

We remind ourselves that we are in the midst of understanding circuits as objects that define relationships between boundary potentials and boundary currents. This relationship is defined by the stipulation that voltage–current pairs on a circuit must obey Ohm’s law and Kirchhoff’s laws—or equivalently, Ohm’s law, Kirchhoff’s voltage law, and the principle of minimum power. In this subsection we show these conditions imply that for each boundary potential $\psi$ on the circuit there exists a potential $\phi$ on the circuit extending $\psi$, unique up to what may be interpreted as a choice of reference potential on each connected component of the circuit. From this potential $\phi$ we can then...
compute the unique voltage, current, and boundary current functions compatible with the given boundary potential.

Fix again a circuit with extended power functional $P : \mathbb{R}^N \to \mathbb{R}$. Let $\nabla : \mathbb{R}^N \to \mathbb{R}^N$ be the operator that maps a potential $\phi \in \mathbb{R}^N$ to the function $N \to \mathbb{R}$ given by

$$ n \mapsto \left. \frac{\partial P}{\partial \phi(n)} \right|_{\phi=0}. $$

As we have seen, this function takes potentials to twice the pointwise currents that they induce. We have also seen that a potential $\phi$ is compatible with the governing laws of circuits if and only if

$$ (\nabla \phi)|_{\partial \phi N} = 0 $$

The operator $\nabla$ acts as a discrete analogue of the Laplacian for the graph $\Gamma$, so we call this operator the Laplacian of $\Gamma$, and say that the equation (1) is a version of Laplace’s equation. We then say that the problem of finding an extension $\phi$ of some fixed boundary potential $\psi$ that solves this Laplace’s equation—or, equivalently, the problem of finding a $\phi$ that obeys the principle of minimum power for $\psi$—is a discrete version of the Dirichlet problem.

As we shall see, this version of the Dirichlet problem always has a solution. However, the solution is not necessarily unique. If we take a solution $\phi$ and some $\alpha \in \mathbb{R}^N$ that is constant on each connected component and vanishes on the boundary of $\Gamma$, it is clear that $\phi + \alpha$ is still an extension of $\psi$ and that $\left. \frac{\partial P}{\partial \phi(n)} \right|_{\phi=\alpha} = \left. \frac{\partial P}{\partial \phi(n)} \right|_{\phi=\phi+\alpha}$, so $\phi + \alpha$ is another solution. We say that a connected component of a circuit touches the boundary if it contains a vertex in $\partial N$. Note that such an $\alpha$ must vanish on all connected components touching the boundary.

With these preliminaries in hand, we can solve the Dirichlet problem:

**Proposition 2.6.** For any boundary potential $\psi \in \mathbb{R}^\partial N$ there exists a potential $\phi$ obeying the principle of minimum power for $\psi$. If we also demand that $\phi$ vanish on every connected component of $\Gamma$ not touching the boundary, then $\phi$ is unique.

**Proof.** For existence, observe that the power is a nonnegative quadratic form, the extensions of $\psi$ form an affine subspace of $\mathbb{R}^N$, and a nonnegative quadratic form restricted to an affine subspace of a real vector space must reach a minimum somewhere on this subspace.

For uniqueness, suppose that both $\phi$ and $\phi'$ obey the principle of minimum power for $\psi$. Let

$$ \alpha = \phi' - \phi. $$

Then

$$ \alpha|_{\partial N} = \phi'|_{\partial N} - \phi|_{\partial N} = \psi - \psi = 0, $$

so $\phi + \lambda \alpha$ is an extension of $\psi$ for all $\lambda \in \mathbb{R}$. This implies that

$$ f(\lambda) := P(\phi + \lambda \alpha) $$

is a smooth function attaining its minimum value at both $t = 0$ and $t = 1$. In particular, this implies that $f'(0) = 0$. But this means that when writing $f$ as a quadratic, the coefficient of $\lambda$ must be 0, so we can write

$$ f(\lambda) = \sum_{e \in E} \frac{1}{r(e)} \left( (\phi + \lambda \alpha)(t(e)) - (\phi + \lambda \alpha)(s(e)) \right)^2 $$

$$ = \sum_{e \in E} \frac{1}{r(e)} \left( (\phi(t(e)) - \phi(s(e))) + \lambda (\alpha(t(e)) - \alpha(s(e))) \right)^2 $$

$$ = \sum_{e \in E} \frac{1}{r(e)} \left( \phi(t(e)) - \phi(s(e)) \right)^2 + \lambda \text{-term} + \lambda^2 \sum_{e \in E} \frac{1}{r(e)} \left( \alpha(t(e)) - \alpha(s(e)) \right)^2 $$

$$ = \sum_{e \in E} \frac{1}{r(e)} \left( \phi(t(e)) - \phi(s(e)) \right)^2 + \lambda^2 \sum_{e \in E} \frac{1}{r(e)} \left( \alpha(t(e)) - \alpha(s(e)) \right)^2. $$
Then
\[ f(1) - f(0) = \sum_{e \in E} \frac{1}{r(e)} \left( \alpha(t(e)) - \alpha(s(e)) \right)^2 = 0, \]
so \( \alpha(t(e)) = \alpha(s(e)) \) for every edge \( e \in E \). This implies that \( \alpha \) is constant on each connected component of the graph \( \Gamma \) of our circuit.

Note that as \( \alpha|_{\partial N} = 0 \), \( \alpha \) vanishes on every connected component of \( \Gamma \) touching the boundary. Thus, if we also require that \( \phi \) and \( \phi' \) vanish on every connected component of \( \Gamma \) not touching the boundary, then \( \alpha = \phi' - \phi \) vanishes on all connected components of \( \Gamma \), and hence is identically zero. Thus \( \phi' = \phi \), and this extra condition ensures a unique solution to the Dirichlet problem.

We have also shown the following.

**Proposition 2.7.** Suppose \( \psi \in \mathbb{R}^{\partial N} \) and \( \phi \) is a potential obeying the principle of minimum power for \( \psi \). Then \( \phi' \) obeys the principle of minimum power for \( \psi \) if and only if the difference \( \phi' - \phi \) is constant on every connected component of \( \Gamma \) and vanishes on every connected component touching the boundary of \( \Gamma \).

Furthermore, \( \phi \) depends linearly on \( \psi \).

**Proposition 2.8.** Fix \( \psi \in \mathbb{R}^{\partial N} \), and suppose \( \phi \in \mathbb{R}^N \) is the unique potential obeying the principle of minimum power for \( \psi \) that vanishes on all connected components of \( \Gamma \) not touching the boundary. Then \( \phi \) depends linearly on \( \psi \).

**Proof.** Fix \( \psi, \psi' \in \mathbb{R}^{\partial N} \), and suppose \( \phi, \phi' \in \mathbb{R}^N \) obey the principle of minimum power for \( \psi, \psi' \) respectively, and that both \( \phi \) and \( \phi' \) vanish on all connected components of \( \Gamma \) not touching the boundary.

Then, for all \( \lambda \in \mathbb{R} \),
\[ (\phi + \lambda \phi')|_{\partial N} = \phi|_{\partial N} + \lambda \phi'|_{\partial N} = \psi + \lambda \psi' \]
and
\[ (\nabla(\phi + \lambda \phi'))|_{\partial N} = (\nabla \phi)|_{\partial N} + \lambda (\nabla \phi')|_{\partial N} = 0. \]
Thus \( \phi + \lambda \phi' \) solves the Dirichlet problem for \( \psi + \lambda \psi' \), and thus \( \phi \) depends linearly on \( \psi \). \( \square \)

Bamberg and Sternberg [7] describe another way to solve the Dirichlet problem, going back to Weyl [60].

### 2.4 Equivalent circuits

We have seen that boundary potentials determine, essentially uniquely, the value of all the electric properties across the entire circuit. But from the perspective of control theory, this internal structure is irrelevant: we can only access the circuit at its terminals, and hence only need concern ourselves with the relationship between boundary potentials and boundary currents. In this section we streamline our investigations above to state the precise way in which boundary currents depend on boundary potentials. In particular, we shall see that the relationship is completely captured by the functional taking boundary potentials to the minimum power used by any extension of that boundary potential. Furthermore, each such power functional determines a different boundary potential–boundary current relationship, and so we can conclude that two circuits are equivalent if and only if they have the same power function.

A ‘behaviour’ is an equivalence class of circuits, where two are considered equivalent when the boundary current is the same function of the boundary potential. The idea is that the boundary current and boundary potential are all that can be observed ‘from outside’, i.e. by making measurements at the terminals. Restricting our attention to what can be observed by making measurements
at the terminals amounts to treating a circuit as a ‘black box’: that is, treating its interior as hidden from view. So, two circuits give the same behaviour when they behave the same as ‘black boxes’.

First let us check that the boundary current is a function of the boundary potential. For this we introduce an important quadratic form on the space of boundary potentials:

**Definition 2.9.** The (scaled) power functional $Q : \mathbb{R}^{\partial N} \to \mathbb{R}$ of a circuit with extended power functional $P$ is given by

$$Q(\psi) = \frac{1}{2} \min_{\phi|_{\partial N} = \psi} P(\phi).$$

Proposition 2.6 shows the minimum above exists, so the power functional is well-defined. Up to a factor of $\frac{1}{2}$, $Q(\psi)$ is just the power dissipated by the circuit when the boundary voltage is $\psi$, thanks to the principle of minimum power. The factor of $\frac{1}{2}$ simplifies the next proposition, which uses $Q$ to compute the boundary current as a function of the boundary voltage. We will later see that in fact $Q(\psi)$ is a nonnegative quadratic form on $\mathbb{R}^{\partial N}$.

Since $Q$ is a smooth real-valued function on $\mathbb{R}^{\partial N}$, its differential $dQ$ at any given point $\psi \in \mathbb{R}^{\partial N}$ is an element of the dual space $(\mathbb{R}^{\partial N})^*$, which we denote by $dQ(\psi)$. In fact, this element is equal to the boundary current $\iota$ corresponding to the boundary voltage $\psi$:

**Proposition 2.10.** Suppose $\psi \in \mathbb{R}^{\partial N}$. Suppose $\phi$ is any extension of $\psi$ minimizing the power. Then $dQ(\psi) \in (\mathbb{R}^{\partial N})^* \cong \mathbb{R}^{\partial N}$ gives the boundary current of the current induced by the potential $\phi$.

**Proof.** Note first that while there may be several choices of $\phi$ minimizing the power subject to the constraint that $\phi|_{\partial N} = \psi$, Proposition 2.7 says that the difference between any two choices vanishes on all components touching the boundary of $\Gamma$. Thus, these two choices give the same value for the boundary current $\iota : \partial N \to \mathbb{R}$. So, with no loss of generality we may assume $\phi$ is the unique choice that vanishes on all components not touching the boundary. Write $\tau : N \to \mathbb{R}$ for the extension of $\iota : \partial N \to \mathbb{R}$ to $N$ taking value 0 on $N \setminus \partial N$.

By Proposition 2.8, there is a linear operator

$$f : \mathbb{R}^{\partial N} \longrightarrow \mathbb{R}^N$$

sending $\psi \in \mathbb{R}^{\partial N}$ to this choice of $\phi$, and then

$$Q(\psi) = \frac{1}{2} P(f \psi).$$
Given any $\psi' \in \mathbb{R}^{\partial N}$, we thus have

$$dQ_\psi(\psi') = \frac{d}{d\lambda} Q(\phi + \lambda \psi') \bigg|_{\lambda=0}$$

$$= \frac{1}{2} \frac{d}{d\lambda} P(f(\psi + \lambda \psi')) \bigg|_{\lambda=0}$$

$$= \frac{1}{2} \frac{d}{d\lambda} \sum_{e \in E} \frac{1}{r(e)} \left( (f(\psi + \lambda \psi'))(t(e)) - (f(\psi + \lambda \psi'))(s(e)) \right)^2 \bigg|_{\lambda=0}$$

$$= \frac{1}{2} \frac{d}{d\lambda} \sum_{e \in E} \frac{1}{r(e)} \left( (f\psi(t(e)) - f\psi(s(e))) + \lambda(f\psi'(t(e)) - f\psi'(s(e))) \right)^2 \bigg|_{\lambda=0}$$

$$= \sum_{e \in E} \frac{1}{r(e)} \left( f\psi(t(e)) - f\psi(s(e))(f\psi'(t(e)) - f\psi'(s(e))) \right)$$

$$= \sum_{e \in E} I(e)(f\psi'(t(e)) - f\psi'(s(e)))$$

$$= \sum_{n \in N} \left( \sum_{t(e) = n} I(e) - \sum_{s(e) = n} I(e) \right) f\psi'(n)$$

$$= \sum_{n \in N} I(n)f\psi'(n)$$

$$= \sum_{n \in \partial N} \iota(n)\psi'(n).$$

This shows that $dQ_\psi^* = \iota$, as claimed.

Note this only depends on $Q$, which makes no mention of the potentials at nonterminals. This is amazing: the way power depends on boundary potentials completely characterises the way boundary currents depend on boundary potentials. In particular, in Part III we shall see that this allows us to define a composition rule for behaviours of circuits.

To demonstrate these notions, we give a basic example of equivalent circuits.

**Example 2.11 (Resistors in series).** Resistors are said to be placed in series if they are placed end to end or, more precisely, if they form a path with no self-intersections. It is well known that resistors in series are equivalent to a single resistor with resistance equal to the sum of their resistances. To prove this, consider the following circuit comprising two resistors in series, with input $A$ and output $C$:

Now, the extended power functional $P : \mathbb{R}^{\{A,B,C\}} \to \mathbb{R}$ for this circuit is

$$P(\phi) = \frac{1}{r_{AB}}(\phi(A) - \phi(B))^2 + \frac{1}{r_{BC}}(\phi(B) - \phi(C))^2,$$

while the power functional $Q : \mathbb{R}^{\{A,C\}} \to \mathbb{R}$ is given by minimisation over values of $\phi(B) = x$:

$$Q(\psi) = \frac{1}{2} \min_{x \in \mathbb{R}} \left( \frac{1}{r_{AB}}(\psi(A) - x)^2 + \frac{1}{r_{BC}}(x - \psi(C))^2 \right).$$

Differentiating with respect to $x$, we see that this minimum occurs when

$$\frac{1}{r_{AB}}(x - \psi(A)) + \frac{1}{r_{BC}}(x - \psi(C)) = 0,$$
and hence when \( x \) is the \( r \)-weighted average of \( \psi(A) \) and \( \psi(C) \):

\[
x = \frac{r_{BC}\psi(A) + r_{AB}\psi(C)}{r_{BC} + r_{AB}}.
\]

Substituting this value for \( x \) into the expression for \( Q \) above and simplifying gives

\[
Q(\psi) = \frac{1}{r_{AB} + r_{BC}} (\psi(A) - \psi(C))^2.
\]

This is also the power functional of the circuit

\[
A \xrightarrow{r_{AB} + r_{BC}} C
\]

and so the circuits are equivalent.

### 2.5 Dirichlet forms

In the previous subsection we claimed that power functionals are quadratic forms on the boundary of the open linear resistive circuit whose behaviour they represent. They comprise, in fact, precisely those quadratic forms known as Dirichlet forms.

**Definition 2.12.** Given a finite set \( S \), a **Dirichlet form** on \( S \) is a quadratic form \( Q : \mathbb{R}^S \to \mathbb{R} \) given by the formula

\[
Q(\psi) = \sum_{i,j} c_{ij}(\psi_i - \psi_j)^2
\]

for some nonnegative real numbers \( c_{ij} \).

Note that we may assume without loss of generality that \( c_{ii} = 0 \) and \( c_{ij} = c_{ji} \); we do this henceforth. Any Dirichlet form is nonnegative: \( Q(\psi) \geq 0 \) for all \( \psi \in \mathbb{R}^S \). However, not all nonnegative quadratic forms are Dirichlet forms. For example, if \( S = \{1, 2\} \), the quadratic form \( Q(\psi) = (\psi_1 + \psi_2)^2 \) is not a Dirichlet form. In fact, the concept of Dirichlet form is vastly more general: such quadratic forms are studied not just on finite-dimensional vector spaces \( \mathbb{R}^S \) but on \( L^2 \) of any measure space. When this measure space is just a finite set, the concept of Dirichlet form reduces to the definition above. For a thorough introduction Dirichlet forms, see the text by Fukushima [20]. For a fun tour of the underlying ideas, see the paper by Doyle and Snell [17].

The following characterizations of Dirichlet forms help illuminate the concept:

**Proposition 2.13.** Given a finite set \( S \) and a quadratic form \( Q : \mathbb{R}^S \to \mathbb{R} \), the following are equivalent:

(i) \( Q \) is a Dirichlet form.

(ii) \( Q(\phi) \leq Q(\phi') \) whenever \( |\phi_i - \phi_j| \leq |\phi'_i - \phi'_j| \) for all \( i,j \).

(iii) \( Q(\phi) = 0 \) whenever \( \phi_i \) is independent of \( i \), and \( Q \) obeys the **Markov property**: \( Q(\phi) \leq Q(\psi) \) when \( \psi_i = \min(\phi_i, 1) \).

**Proof.** See Fukushima [20].

While the extended power functionals of circuits are evidently Dirichlet forms, it is not immediate that all power functionals are. For this it is crucial that the property of being a Dirichlet form is preserved under minimising over linear subspaces of the domain that are generated by subsets of the given finite set.
Proposition 2.14. If $Q : \mathbb{R}^{S+T} \rightarrow \mathbb{R}$ is Dirichlet, then
\[
\min_{\nu \in \mathbb{R}^T} Q(-,\nu) : \mathbb{R}^S \rightarrow \mathbb{R}
\]
is Dirichlet.

Proof. We first note that $\min_{\nu \in \mathbb{R}^T} Q(-,\nu)$ is a quadratic form. Again, $\min_{\nu \in \mathbb{R}^T} Q(-,\nu)$ is well-defined as a nonnegative quadratic form also attains its minimum on an affine subspace of its domain. Furthermore $\min_{\nu \in \mathbb{R}^T} Q(-,\nu)$ is itself a quadratic form, as the partial derivatives of $Q$ are linear, and hence the points at with these minima are attained depend linearly on the argument of $\min_{\nu \in \mathbb{R}^T} Q(-,\nu)$.

Now by Proposition 2.13, $Q(\phi) \leq Q(\phi')$ whenever $|\phi_i - \phi_j| \leq |\phi'_i - \phi'_j|$ for all $i,j \in S + T$. In particular, this implies $\min_{\nu \in \mathbb{R}^T} Q(\psi,\nu) \leq \min_{\nu \in \mathbb{R}^T} Q(\psi',\nu)$ whenever $|\psi_i - \psi_j| \leq |\psi'_i - \psi'_j|$ for all $i,j \in S$. Using Proposition 2.13 again then implies that $\min_{\nu \in \mathbb{R}^T} Q(-,\nu)$ is a Dirichlet form. \qed

Corollary 2.15. Let $Q : \mathbb{R}^\partial N \rightarrow \mathbb{R}$ be the power functional for some circuit. Then $Q$ is a Dirichlet form.

Proof. The extended power functional $P$ is a Dirichlet form, and writing $\mathbb{R}^N = \mathbb{R}^\partial N \oplus \mathbb{R}^N \setminus \partial N$ allows us to write
\[
Q(-) = \min_{\phi_0 \in \mathbb{R}^N \setminus \partial N} \frac{1}{2} P(-,\phi_0).
\]

The converse is also true: simply construct the circuit with set of vertices $X$ and an edge of resistance $\frac{1}{c_{ij}}$ between any $i,j \in X$ such that the term $c_{ij}(\psi_i - \psi_j)$ appears in the Dirichlet form. This gives:

Proposition 2.16. A function $Q$ is the power functional for some circuit if and only if $Q$ is a Dirichlet form.

Note that this is an expression of the star-mesh transform, a well-known fact of electrical engineering stating that every circuit of linear resistors is equivalent to some complete graph of resistors between its terminals. For more details see [35]. We may interpret the proof of Proposition 2.14 as showing that intermediate potentials at minima depend linearly on boundary potentials, in fact a weighted average, and that substituting these into a quadratic form still gives quadratic form.

In summary, in this section we have shown the existence of a surjective function
\[
\begin{cases}
\text{circuits of linear resistors} \\
\text{with boundary } \partial N
\end{cases}
\longrightarrow
\begin{cases}
\text{Dirichlet forms on } \partial N
\end{cases}
\]
map two circuits to the same Dirichlet form if and only if they are behaviourally equivalent. In the next section we extend this result to encompass inductors and capacitors too.

3 Inductors and capacitors

The intuition gleaned from the study of resistors carries over to inductors and capacitors too, to provide a framework for studying what are known as passive linear networks. To understand inductors and capacitors in this way, however, we must introduce a notion of time dependency and subsequently the Laplace transform, which allows us to work in the so-called frequency domain. Here, like resistors, inductors and capacitors simply impose a relationship of proportionality between the voltages and currents that run across them. The constant of proportionality is known as the impedence of the component.

As for resistors, the interconnection of such components may be understood, at least formally, as a minimisation of some quantity, and we may represent the behaviours of this class of circuits with a more general idea of Dirichlet form. We conclude this section by noting an obstruction to building a composition rule for Dirichlet forms, motivating our work in Part II.
3.1 The frequency domain and Ohm’s law revisited

In broadening the class of electrical circuit components under examination, we find ourselves dealing with components whose behaviours depend on the rates of change of current and voltage with respect to time. We thus now consider time-varying voltages \( v(t) : [0, \infty) \rightarrow \mathbb{R} \) and currents \( i(t) : [0, \infty) \rightarrow \mathbb{R} \), where \( t \in [0, \infty) \) is a real variable representing time. For mathematical reasons, we restrict these voltages and currents to only those with (i) zero initial conditions (that is, \( f(0) = 0 \)) and (ii) Laplace transform lying in the field

\[
\mathbb{R}(s) = \left\{ Z(s) = \frac{P(s)}{Q(s)} \mid P(s), Q(s) \text{ polynomials over } \mathbb{R} \text{ in } s, Q(s) \neq 0 \right\}
\]

of real rational functions of one variable. While it is possible that physical voltages and currents might vary with time in a more general way, we restrict to these cases as the rational functions are, crucially, well-behaved enough to form a field, and yet still general enough to provide arbitrarily close approximations to currents and voltages found in standard applications.

An inductor is a two-terminal circuit component across which the voltage is proportional to the rate of change of the current. By convention we draw this as follows, with the inductance \( L \) the constant of proportionality:

\[
\begin{array}{c}
\text{---} \\
\text{L}
\end{array}
\]

Writing \( v_L(t) \) and \( i_L(t) \) for the voltage and current varying over time \( t \) across this component respectively, and using a prime \( ' \) to denote derivative with respect to time \( t \), we thus have the relationship

\[
v_L(t) = Li_L'(t).
\]

Permuting the roles of current and voltage, a capacitor is a two-terminal circuit component across which the current is proportional to the rate of change of the voltage. We draw this as follows, with the capacitance \( C \) the constant of proportionality:

\[
\begin{array}{c}
\text{---} \\
\text{C}
\end{array}
\]

Writing \( v_C(t) \), \( i_C(t) \) for the voltage and current across the capacitor, this gives the equation

\[
i_C(t) = Cv_C'(t).
\]

We assume here that inductances \( L \) and capacitances \( C \) are positive real numbers.

Although inductors and capacitors impose a linear relationship if we involve the derivatives of current and voltage, to mimic the above work on resistors we wish to have a constant of proportionality between functions representing the current and voltage themselves. Many integral transforms perform just this role; we choose the Laplace transform. This lets us write a function of time \( t \) instead as a function of frequencies \( s \), and in doing so turns differentiation with respect to \( t \) into multiplication by \( s \), and integration with respect to \( t \) into division by \( s \).

To supply detail, given a function \( f(t) : [0, \infty) \rightarrow \mathbb{R} \), we define the Laplace transform of \( f \)

\[
\mathcal{L}\{f(t)\}(s) = \int_0^\infty f(t)e^{-st}dt.
\]

We also use the notation \( \mathcal{L}\{f\}(s) = F(s) \), denoting the Laplace transform of a function in upper case, and refer to the Laplace transforms as lying in the frequency domain or \( s \)-domain. For us, the three crucial properties of the Laplace transform are then:

\(^4\)We follow the standard convention of denoting inductance by the letter \( L \), after the work of Heinrich Lenz and to avoid confusion with the \( I \) used for current.
(i) linearity: $\mathcal{L}\{af + bg\}(s) = aF(s) + bG(s)$;

(ii) differentiation: $\mathcal{L}\{f'(t)\}(s) = sF(s) - f(0)$;

(iii) integration: $\mathcal{L}\{\int_0^t f(\tau) d\tau\} = \frac{1}{s}F(s)$.

Writing $V(s)$ and $I(s)$ for the Laplace transform of the voltage $v(t)$ and current $i(t)$ across a component respectively, and recalling that by assumption $v(t) = i(t) = 0$, the $s$-domain behaviours of components become, for a resistor of resistance $R$:

$$V(s) = RI(s)$$

for an inductor of inductance $L$:

$$V(s) = sLI(s)$$

and for a capacitor of capacitance $C$:

$$V(s) = \frac{1}{sC}I(s)$$

In particular, for each component the voltage $V$ is proportional to the current $I$ with constant of proportionality lying in the subset

$$\mathbb{R}(s)^+ = \left\{Z(s) = \frac{P(s)}{Q(s)} \in \mathbb{R}(s) \mid Z(x) \geq 0 \text{ for all } x \geq 0, Q(x) \neq 0\right\}$$

of $\mathbb{R}(s)$ comprising rational functions that are positive on the positive real axis. We call this constant the impedance, and denote it by $Z$. That is, the impedance of a two-terminal passive linear circuit is the frequency-domain ratio of the voltage to the current with zero initial conditions.

We define a passive linear circuit to be a circuit over $\mathbb{R}(s)^+$. It is convention to draw a component of these circuits, one with impedance $Z$, as

\[ \begin{array}{c}
\bullet \\
Z \\
\bullet
\end{array} \]

The behaviour of such a component is given by the $s$-domain generalisation of Ohm’s law:

$$V(s) = Z(s)I(s).$$

### 3.2 Generalised Dirichlet forms

To understand the behaviour of passive linear circuits, we also need to understand how the behaviours of individual components, governed by Ohm’s law, fit together give the behaviour of an entire network. This is still given by Kirchhoff’s laws.

In particular, although we now work over the field $\mathbb{R}(s)$ instead of $\mathbb{R}$, an analogue of Proposition 2.5 still holds. In the following, since many objects are functions of both $s$ and $e \in E$ or $n \in N$, we shall indicate some arguments in subscript to avoid unwieldy notation. For example, for a voltage $V : \mathbb{R}(s)^E \to \mathbb{R}$ we shall denote the voltage across some edge $e$ as $V_e$ or $V_e(s)$.

In analogy with the extended power functional for resistor networks, define the $s$-domain power functional $P : \mathbb{R}(s)^N \to \mathbb{R}(s)$ of a circuit to be the map

$$P(\varphi) = \sum_{e \in E} \frac{1}{Z_e} \left(\varphi(t(e)) - \varphi(s(e))\right)^2.$$

Although it is not clear what it means to minimise over the field $\mathbb{R}(s)$, we can use formal derivatives to formulate an analogue, sans physical interpretation of power minimisation, of the principle of
minimum power. Indeed, the power functional \( P(\varphi) \) can be considered an element of the polynomial ring \( \mathbb{R}(s)[\{\varphi(n)\}_{n \in N}] \), considering the \( \varphi(n) \) as formal variables, where \( n \in N \) are the vertices of the circuit. We may thus take formal derivatives with respect to the \( \varphi(n) \), with the derivative of
\[
\frac{\partial P}{\partial \varphi(n)} = ka_k \varphi(n)^{k-1} + \cdots + 2a_2 \varphi(n) + a_1 \in \mathbb{R}(s)[\{\varphi(n)\}_{n \in N}],
\]
where \( a_i \in \mathbb{R}(s)[\{\varphi(m)\}_{m \in N \setminus \{n\}}] \), and where \( ia_i \) is given by the sum \( a_i + \cdots + a_i \) with \( i \) summands. We then call a potential \( \phi \in \mathbb{R}(s)^N \) a **realisable potential** of a circuit with power functional \( P \) if for each interior node \( n \in N \setminus \partial N \) the formal partial derivative of the power functional with respect to \( \varphi(n) \) evaluated at \( \phi \) is equal to zero:
\[
\frac{\partial P}{\partial \varphi(n)} \bigg|_{\varphi=\phi} = 0
\]
This terminology arises from the following fact, the analogue of Proposition 2.5 for passive linear circuits:

**Proposition 3.1.** The potential \( \phi \in \mathbb{R}(s)^N \) is a realisable potential of a given passive linear circuit if and only if the induced current \( I_e = \frac{1}{Z_e}(\phi(t(e)) - \phi(s(e))) \) obeys Kirchhoff’s current law.

The proof of this statement is exactly that for Proposition 2.5. Note that it follows from Proposition 2.6 that, for a circuit containing only resistors and whose boundary intersects every connected component, the realisable potentials are in one-to-one correspondence with boundary potentials.

A corollary of Proposition 3.1 is that the set of states—that is, potential–current pairs—that are compatible with the governing laws of a circuit is given by the set of realisable potentials together with their induced currents. This the behaviour of a circuit may be specified precisely by their \( s \)-domain power functional. The form of these functionals motivates us to make the following definition.

**Definition 3.2.** Let \( F \) be a field, and let \( F^+ \) be a subrig of \( F \). Given a finite set \( S \), a **Dirichlet form** over \( F \) on \( S \) is a quadratic form \( Q : F^S \rightarrow F \) given by the formula
\[
Q(\psi) = \sum_{i,j \in S} c_{ij}(\psi_i - \psi_j)^2
\]
for \( c_{ij} \in F^+ \).

In particular, we may note that power functionals of passive linear circuits are Dirichlet forms over \( \mathbb{R}(s) \). In this setting too, the behaviour of a circuit is thus again given by the potentials and corresponding pointwise currents for which the Dirichlet form, or power functional, of the circuit has zero formal derivative on each of the internal nodes. Moreover, such states can again be constructed from boundary potentials. Indeed, we have the following analogue of Proposition 2.14.

**Proposition 3.3.** Let \( Q(\psi) = \sum_{i,j} c_{ij}(\psi_i - \psi_j)^2 \) be a Dirichlet form over \( F \) on \( S \), and let \( s \in S \) be an element of \( S \). Then the function \( \min_s Q : F^{S \setminus \{s\}} \rightarrow F \),
\[
\min_s Q(\psi) = \sum_{i,j} c_{ij}(\psi_i - \psi_j)^2 + \sum_{\ell} c_{\ell s} \left( \psi_{\ell} - \frac{\sum_k c_{ks} \psi_k}{\sum_k c_{ks}} \right)^2,
\]
is a Dirichlet form on \( S \setminus \{s\} \).

Note that in the above expression sums are taken over \( S \setminus \{s\} \), and without loss of generality we have assumed that \( c_{sk} = 0 \) for all \( k \), so we need only consider the coefficients \( c_{ks} \).
This construction is given by, for each \( \psi \in \mathbb{F}^{S \setminus \{s\}} \), choosing \( \tilde{\psi}_s \in \mathbb{F} \) such that
\[
\frac{\partial Q}{\partial \varphi(s)}|_{\varphi = \tilde{\psi}} = 0,
\]
where \( \tilde{\psi} \in \mathbb{F}^S \) is equal to \( \psi \) on the set \( S \setminus \{s\} \) and takes value \( \psi_s \) at \( s \). Indeed,
\[
\left. \frac{\partial Q}{\partial \varphi(s)} \right|_{\varphi = \psi} = \sum_k 2c_{ks}(\psi_s - \psi_k)
\]
is equal to zero when
\[
\psi_s = \sum_k c_{ks} \tilde{\psi}_k \sum_k c_{ks}.
\]

Proof. As the sum of Dirichlet forms is evidently Dirichlet, it suffices to check that the expression
\[
\sum_{\ell} c_{\ell s} \left( \psi_\ell - \sum_k c_{ks} \tilde{\psi}_k \right)^2
\]
is Dirichlet on \( S \setminus \{s\} \). Multiplying through by the constant \( (\sum_k c_{ks})^2 \in \mathbb{F}^+ \), it further suffices to check
\[
\sum_{\ell} c_{\ell s} \left( \sum_k c_{ks} \psi_\ell - \sum_k c_{ks} \tilde{\psi}_k \right)^2 = \sum_{\ell} c_{\ell s} \left( \sum_k c_{ks} (\psi_\ell - \tilde{\psi}_k) \right)^2
\]
\[
= \sum_{\ell} c_{\ell s} \left( \sum_{k, m} c_{ks} c_{ms} (\psi_\ell - \tilde{\psi}_k)(\psi_\ell - \tilde{\psi}_m) + \sum_k c_{ks}^2 (\psi_\ell - \tilde{\psi}_k)^2 \right)
\]
\[
= 2 \sum_{k, \ell, m \neq m} c_{ks} c_{ks} (\psi_\ell - \tilde{\psi}_k)(\psi_\ell - \tilde{\psi}_m) + \sum_{k, \ell} c_{\ell s} c_{ks}^2 (\psi_\ell - \tilde{\psi}_k)^2
\]
is Dirichlet. But
\[
(\psi_k - \psi_\ell)(\psi_k - \psi_m) + (\psi_\ell - \tilde{\psi}_k)(\psi_\ell - \tilde{\psi}_m) + (\tilde{\psi}_m - \psi_k)(\psi_m - \psi_\ell)
\]
\[
= \psi_k^2 + \psi_\ell^2 + \psi_m^2 - \psi_k \psi_\ell - \psi_k \tilde{\psi}_m - \psi_m \psi_\ell
\]
\[
= \frac{1}{2} ((\psi_k - \psi_\ell)^2 + (\psi_k - \psi_m)^2 + (\psi_\ell - \psi_m)^2),
\]
so this expression is indeed Dirichlet. Indeed, pasting these computations together shows that
\[
\min_s Q(\psi) = \sum_{i, j} \left( c_{ij} + \frac{c_{is} c_{js}}{\sum_k c_{ks}} \right) (\psi_i - \psi_j)^2.
\]

As formal partial derivatives commute, this process may be iterated to define, for any subset \( R \) of \( S \), the Dirichlet form \( \min_R Q \) on \( S \setminus R \). In the next section we describe an attempt at constructing a composition rule for Dirichlet forms based on this fact.

### 3.3 Composition of Dirichlet forms

To motivate the subsequent part of this paper, we illustrate a naïve attempt to construct a category where the morphisms are behaviours of circuits. This naïve attempt doesn’t quite work, because it
doesn’t include identity morphisms. However, it points in the right direction, and underlines the
importance of the cospan formalise we then move on to develop.

Recall that behaviours, or equivalence classes, of circuits can be specified by Dirichlet forms over
\[ \mathbb{R}(s) \], and in fact that resistive circuits are in one-to-one correspondence with Dirichlet forms over \( \mathbb{R} \).
We define a composition rule for Dirichlet forms that reflects composition of circuits. Given finite
sets \( S \) and \( T \), let \( S + T \) denote their disjoint union. Let \( D(S,T) \) be the set of Dirichlet forms on
\( S + T \). There is a way to compose these Dirichlet forms
\[ \circ : D(T,U) \times D(S,T) \to D(S,U) \]
defined as follows. Given \( Q \in D(S,T) \) and \( R \in D(T,U) \), let
\[ (R \circ Q)(\gamma,\alpha) = \min_{T} Q(\gamma,\beta) + R(\beta,\alpha), \]
where \( \alpha \in R^S, \gamma \in R^U \). This operation has a clear interpretation in terms of electrical circuits: the
power used by the entire circuit is just the sum of the power used by its parts.

It is immediate from Proposition 3.3 that this composition rule is well-defined: the composite of
two Dirichlet forms is again a Dirichlet form. Moreover, this composition is associative:
\[ (P \circ Q) \circ R = P \circ (Q \circ R). \]

However, it fails to provide the structure of a category as there is typically no Dirichlet form
\( 1_S \in D(S,S) \) playing the role of the identity for this composition. For an indication of why this is
so, consider the case of resistive circuits, and let \( \{\bullet\} \) be a set with one element, and suppose that
\( 1_{\{\bullet\}}(\alpha,\beta) = k(\alpha - \beta)^2 \in D(\{\bullet\},\{\bullet\}) \) acts as an identity on the right for this composition. Then for
all \( Q(\alpha,\beta) = c(\alpha - \beta)^2 \in D(\{\bullet\},\{\bullet\}) \), we must have
\[
ca^2 = Q(\alpha,0) \\
= (Q \circ 1_{\{\bullet\}})(\alpha,0) \\
= \min_{\beta \in \mathbb{R}^{\{\bullet\}}} 1_{\{\bullet\}}(\alpha,\beta) + Q(\beta,0) \\
= \min_{\beta \in \mathbb{R}^{\{\bullet\}}} k(\alpha - \beta)^2 + c\beta^2 \\
= \frac{k}{k+c} \alpha^2,
\]
where we have checked that the \( k(\alpha - \beta)^2 + c\beta^2 \) is minimised with respect to \( \beta \) when its derivative
with respect to \( \beta \) is zero. But for all values of \( k \in \mathbb{R} \) this is true only when \( c = 0 \), so no such
Dirichlet form exists. Note, however, that for \( k \gg c \) we have \( ca^2 \approx \frac{k}{k+c} \alpha^2 \), so Dirichlet forms with
large values of \( k \)—corresponding to resistors with resistance close to zero—act as ‘almost’ identities.
In this way we might interpret the identities we wish to introduce into this category as idealised
components with zero resistance.

Nonetheless, we have most of the structure required for a category. A ‘category without identity
morphisms’ is called a semicategory, so we see

**Proposition 3.4.** There is a semicategory where:

- the objects are finite sets,
- a morphism from \( T \) to \( S \) is a Dirichlet form \( Q \in D(S,T) \).
- composition of morphisms is given by
\[
(R \circ Q)(\gamma,\alpha) = \min_{\beta \in \mathbb{R}^T} Q(\gamma,\beta) + R(\beta,\alpha).
\]
We would like to make this into a category. One easy way to do this is to formally adjoin identity morphisms; this trick works for any semicategory. However, we obtain a better category if we include \textit{more} morphisms: more circuits having wires with zero impedance. As the expression for the extended power functional includes the reciprocals of impedances, such circuits cannot be expressed within the framework we have developed thus far. Indeed, for these idealised circuits there is no function taking boundary potentials to boundary currents: the lack of impedance is thought as as implying that any difference in potentials at the boundary induces ‘infinite’ currents. To accommodate this idea, we introduce sympletic vector spaces and their Lagrangian subspaces. We first, however, develop a category theoretic framework, based around decorated cospans, to orient ourselves within. The first thing this will allow us to do is provide a definition of the category of circuits itself, and an understanding of its basic properties.

Part II
Categories of Circuits

In this part we move our focus from the semantics of circuit diagrams to the syntax, addressing the question “How do we interact with circuit diagrams?”. Informally, the answer to this is that we interact with them by connecting them to each other, perhaps after moving them into the right form by rotating or reflecting them, or by crossing or bending some of the wires. To formalise this, we adopt a category theoretic viewpoint, defining various dagger compact categories with circuits as morphisms. We claim a formal analysis of this structure, especially of the composition or connection of circuits, has been overlooked in analysis of circuits thus far. This part culminates in the definition of two important categories, the category Circ of circuit diagrams, and the category LagrRel containing all behaviours of circuits. We also develop the technical material required to appreciate the structure of these categories, and that aids understanding of the relationship between the two, to be addressed in Part III.

4 Decorated cospans

We begin now with a technical section describing a general technique for developing composition rules for structures on finite sets. As we have seen, whether represented by circuit diagrams or Dirichlet forms, circuits can be described as structures on a finite set of nodes. While this provides a good classification of the different types of circuits that exist, it does not allow for discussion of their composition. In this section, however, we describe a method for taking (1) a description of a structure that can be placed on finite sets together with (2) a description of how this structure interacts with functions between these sets, and producing a category which describes composition of structures. This category is built as a cospan category, with the apex of the cospan describing some structure, such as a circuit, and the feet of the cospan describing possible interfaces to this structure.

4.1 Cospan categories

Recall that a \textit{cospan} from $X$ to $Y$ in a category $\mathcal{C}$ is an object $C$ in $\mathcal{C}$ with a pair of morphisms $f : X \to C$, $g : Y \to C$:
We shall refer to \(X\) and \(Y\) as the **feet**, and \(C\) as the **apex** of the cospan. When such pushouts exist, cospans may be composed using the pushout from the common foot: given cospans \(X \xleftarrow{f} C \xrightarrow{g} Y\) from \(X\) to \(Y\) and \(Y \xleftarrow{f'} C' \xrightarrow{g'} Z\) from \(Y\) to \(Z\), their composite cospan is \(X \xleftarrow{i \circ f} P \xrightarrow{i' \circ g'} Z\) where \(P, i : C \to P\), and \(i' : C' \to P\) form the top half of the pushout square:

\[
\begin{array}{ccc}
X & \xleftarrow{f} & C \\
& g & \downarrow \downarrow \\
Y & \xleftarrow{f'} & C'
\end{array}
\quad \begin{array}{ccc}
P & \xleftarrow{i} & C \\
& i' & \downarrow \downarrow \\
Z & \xleftarrow{g'} & C'
\end{array}
\]

A **map of cospans** is a morphism \(h : C \to C'\) in \(\mathcal{C}\) between the apices of two cospans \(X \xleftarrow{f} C \xrightarrow{g} Y\) and \(X \xleftarrow{f'} C' \xrightarrow{g'} Y\) with the same feet, such that

\[
\begin{array}{ccc}
X & \xleftarrow{f} & C \\
& h & \downarrow \downarrow \\
Y & \xleftarrow{f'} & C'
\end{array}
\]

commutes. Given a category \(\mathcal{C}\) with pushouts, we may thus define a category \(\text{Cospan}(\mathcal{C})\) with objects the objects of \(\mathcal{C}\) and morphisms isomorphism classes of cospans. We will often abuse our terminology and refer to cospans themselves as morphisms in some cospan category \(\text{Cospan}(\mathcal{C})\); we of course refer instead to the isomorphism class of the said cospan. Note that there always exists a wide embedding

\[
\mathcal{C} \hookrightarrow \text{Cospan}(\mathcal{C}),
\]

with the embedding functor taking objects of \(\mathcal{C}\) to their corresponding object in \(\text{Cospan}(\mathcal{C})\), and taking morphisms \(f : X \to Y\) to the cospan \(X \xleftarrow{f} Y \xrightarrow{1_Y} Y\). For this reason we often refer to \(\mathcal{C}\) as a (wide) subcategory of \(\text{Cospan}(\mathcal{C})\). We also note that such cospan categories come equipped with a so-called dagger functor, which maps a cospan \(X \xleftarrow{f} C \xrightarrow{g} Y\) to its reflection \(Y \xleftarrow{g} C \xrightarrow{f} X\).

### 4.2 Dagger compact categories

Recall that a **dagger functor** is an involutive, contravariant endofunctor that is the identity on objects. That is, given a category \(\mathcal{C}\), a dagger functor is a contravariant functor \(\dagger : \mathcal{C} \to \mathcal{C}\) such that \(\dagger(A) = A\) for all objects \(A \in \text{Ob}\mathcal{C}\) and \(\dagger(\dagger(f)) = f\) for all morphisms \(f\) in \(\mathcal{C}\). A dagger functor expresses the idea that the direction of morphisms can be reversed: through a dagger functor each morphism specifies a map from its codomain to its domain, in addition to the map it is from its domain to its codomain. This is true of electrical circuits: if we like we may treat the set of inputs as the set of outputs instead, and the set of outputs as the set of inputs.

When other structure is present, we prefer this dagger to play nice with it. A **dagger symmetric monoidal category** is a symmetric monoidal category equipped with a dagger symmetric monoidal functor—that is, a dagger functor that coherently preserves the symmetric monoidal structure. Concretely, this requires that the dagger functor preserve the tensor product, and that the associator, unitors, and braiding of the symmetrical monoidal category be unitary. Furthermore, letting \(L\) and \(R\) be dual objects of a dagger symmetric monoidal category, with monoidal unit \(I\), braiding
\( \sigma_{L,R} : L \otimes R \to R \otimes L \), and unit \( \eta : I \to R \otimes L \) and counit \( \epsilon : L \otimes R \to I \), we say that \( L \) and \( R \) are \textbf{dagger dual} if \( \eta = \sigma \circ \epsilon^\dagger \). A \textbf{dagger compact category} is a dagger symmetric monoidal category in which every object has a dagger dual.

Importantly for our applications, dagger compact categories come with a graphical calculus, where each morphism is represented by a diagram such that two diagrams are considered equal by the rules of this calculus if and only if they are equal according to the defining laws of dagger compact categories [48]. In brief, to set up our conventions, we represent a morphism \( f : X_1 \otimes \cdots \otimes X_n \to Y_1 \otimes \cdots \otimes Y_m \) as a ‘downwards flow chart’:

![Downwards flow chart](image)

Composition is then represented by connecting the lines (or wires) representing the codomain of one morphism with the domain of the another placed below it, the monoidal product of two morphisms is represented by their side-by-side juxtaposition, the swap map by crossing two wires, the compact structure by bending a wire 180 degrees, and the dagger functor by flipping a diagram in the horizontal axis. We believe these operations on diagrams—placing diagrams on the same page, rearranging their wires/flipping them, and then connecting their wires to form a larger diagram—represents the collection of operations used for reasoning with circuit diagrams, and hence that dagger compact categories are an appropriate structure for the formalisation of such.

**Example 4.1.** Of particular interest is the category \( \text{Cospan}(\text{FinSet}) \) of cospans in \( \text{FinSet} \), the category of finite sets and functions. We think of this category as a dagger compact category, with the monoidal product given by coproducts in the category of finite sets, the symmetric monoidal structure maps inherited from viewing \( (\text{FinSet}, +) \) as a wide subcategory, the dagger functor the aforementioned, and each finite set \( X \) dual to any isomorphic set \( \overline{X} \) with unit and counit

![Cospan diagram](image)

respectively, where \( ! \) is the unique function of the given type, \( x : \overline{X} \to X \) is an isomorphism, and we write \([f, g] \) for the coproduct of morphisms \( f \) and \( g \). It is a simple computation to check that this monoidal product is functorial, and that the dagger functor interacts with the symmetric monoidal structure and the duals for objects to indeed define a dagger compact category. Note that the monoidal product here is the coproduct, or disjoint union, of sets. This decision is informed by the application to circuit theory.

We will spend much time discussing categories built from \( \text{Cospan}(\text{FinSet}) \). In the next subsection we describe methods to construct such categories.

### 4.3 Decorated cospan categories

While cospans in \( \text{FinSet} \) provide a good language to describe connections between finite sets—we view the feet as indexing connection points of the apex—to provide a framework for describing composition of more interesting structures than finite sets we need a way of incorporating this extra data. This is provided by the idea of an \( F \)-decorated cospan: a cospan in \( \text{FinSet} \) in which the apex \( N \) is equipped with an element \( 1 \to FN \) of some object \( FN \). We think of \( F \) as describing
the collection of available structures on $N$, with examples such as the circuit of circuit diagrams or Dirichlet forms on $N$ in mind.

We show that when $F$ forms a lax monoidal functor, then such decorated cospans form a category.

**Lemma 4.2.** Let 

$$(F, \varphi) : (\text{FinSet}, +) \rightarrow (\mathcal{C}, \otimes)$$

be a lax monoidal functor. We may define a category $\mathbf{FCospan}$, the category of $F$-decorated cospans, with objects finite sets, and morphisms equivalence classes of pairs

$$(X \xrightarrow{i} N \leftarrow Y, \ 1 \xrightarrow{s} FN)$$

comprising a cospan $X \xrightarrow{i} N \leftarrow Y$ in $\text{FinSet}$ together with an element $1 \xrightarrow{s} FN$ of the $F$-image $FN$ of the apex of the cospan. We shall call the element $1 \xrightarrow{s} FN$ the **structure element** of the decorated cospan. Equivalence is defined up to isomorphism of cospans; an isomorphism of cospans induces a one-to-one correspondence between structure elements of their apices.

Composition in this category is given by pushout of cospans in $\text{FinSet}$

paired with the pushforward

$$1 \xrightarrow{s^{-1}} 1 \otimes 1 \xrightarrow{\varphi_{N,M}} FN \otimes FM \xrightarrow{\varphi_{N,M}} F(N + M) \xrightarrow{F[j_N, j_M]} F(N +_Y M)$$

of the tensor product of the structure elements along the coproduct of the pushout maps.

The key insight of the construction is contained in this last sentence: although at first glance it might seem surprising that we can construct a composition rule for structure elements $s : 1 \rightarrow FN$ and $t : 1 \rightarrow GM$ just from an understanding of how structure elements pushforward along functions between their underlying sets, the monoidality of $F$ means we can take the product of their structure elements $s \otimes t$, up to isomorphism an element of $F(N + M)$, and then push it along the coproduct of the pushout maps, a function $N + M \rightarrow N +_Y M$, to get an element of the $F(N +_Y M)$. This pushforward encodes the identification of the image of $Y$ in $N$ with the image of the same in $M$, and so describes the ‘connecting’ of the two ‘circuits’.

**Proof of Lemma 4.2.** We check that we have a well-defined category. For this we need to check that this composition rule is associative, and that each object has an identity morphism.

**Associativity:** Suppose we have morphisms

$$(X \xrightarrow{i_X} N \leftarrow^o Y, \ 1 \xrightarrow{s} FN),$$

$$(Y \xrightarrow{i_Y} M \leftarrow^o Z, \ 1 \xrightarrow{t} FM),$$

$$(Z \xrightarrow{i_Z} P \leftarrow^o W, \ 1 \xrightarrow{u} FP).$$
As pushouts are unique up to unique isomorphism, we know that the composite of these three cospans in FinSet is associative. We must check that the pushforward of the elements is also an associative process. Write

\[ \tilde{\alpha} : (N +_Y M) +_Z P \rightarrow N +_Y (M +_Z P) \]

for the unique isomorphism between the two pairwise pushouts constructions from the above three cospans. Consider then the following diagram, with top row the element obtained by taking the composite of the first two morphisms first, and the bottom row the element obtained by taking the composite of the last two morphisms first.

This diagram commutes as (1) is the triangle coherence equation for the monoidal category \((\mathcal{C}, \otimes)\), (2) is naturality for the associator \(\alpha\), (3) is the associativity condition for the monoidal

\[ 1 \otimes 1 \]

\[ \chi^{-1} \]

\[ 1 \]

\[ (s \otimes t) \otimes u \]

\[ s \otimes (t \otimes u) \]

\[ (1 \otimes 1) \otimes 1 \]

\[ 1 \otimes (1 \otimes 1) \]

\[ \rho^{-1} \otimes 1 \]

\[ 1 \otimes \lambda^{-1} \]
functor \( F \), (4) and (5) commute by the naturality of \( \varphi \), and (6) commutes as it is the \( F \)-image of a hexagon describing the associativity of the pushout. This shows that the two structure elements obtained by the two different orders of composition of our three morphisms are equal up to the unique isomorphism \( \tilde{\alpha} \) between the two different pushouts that may be obtained. Our composition rule is hence associative.

**Identity morphisms:** The morphism \( X \to X \) given by the pair

\[
(X \xrightarrow{1_X} X \xleftarrow{1_X} X, \ 1 \xrightarrow{F! \circ \varphi_1} FN)
\]

acts as an identity for the composition we have defined, where \( \varphi_1 : 1 \to F\emptyset \) is the unit coherence map for the monoidal functor. We shall show that it is an identity for composition on the left; the case for composition on the right is similar. Observe that the cospan in this pair is known to be the identity cospan in FinSet. We thus need to check that, given a morphism

\[
(X \xrightarrow{i} N \xleftarrow{a} Y, \ 1 \xrightarrow{s} FN),
\]

the pushforward of the product \((F! \circ \varphi_1) \otimes s\) along the \( F \)-image of the coproduct \([i_X, 1_N] : X+N \to N\) of the pushout maps is again the element \( s\); this pushforward being, by definition, the structure element of the composite of the given morphism and the claimed identity map. This is shown by the commutativity of the diagram below, with the path along the lower edge equal to the aforementioned pushforward.

This diagram commutes as each subdiagram commutes: (1) commutes by the naturality of \( \lambda \), (2) by the unit monoidality of the functor \( F \), (3) by the interchange law, (4) by the naturality of \( \varphi \), and (5) as it is the \( F \)-image of the commutative triangle

\[
\emptyset + N \xrightarrow{[i_X, 1_N]} N
\]

in FinSet. □

Decorated cospan categories are so named as they generalise the category of cospans of finite sets.
Lemma 4.3. Let \((F, \varphi) : (\text{FinSet}, +) \to (\mathcal{C}, \otimes)\) be a lax monoidal functor. Then there is a wide embedding

\[ \text{Cospan}(\text{FinSet}) \hookrightarrow \text{FCosp}an. \]

Proof. Given a finite set \(N\), call \(1 \xrightarrow{\varphi_1} F\emptyset \xrightarrow{F1} FN\) the empty structure on \(N\). We then let this embedding send finite sets to themselves, and decorate cospan \(X \xrightarrow{i} N \xleftarrow{o} Y\) with the empty structure, giving the decorated cospan \((X \xrightarrow{i} N \xleftarrow{o} Y, 1 \xrightarrow{F1\varphi_1} FN)\). This functor, should it be well-defined, is evidently bijective on objects and faithful.

To check that this functor is well-defined, we must check the composite of empty structures is again an empty structure. This involves a now routine diagram chase of the above sort, based on the observation that the monoidality of the functor \(F\) allows us to factor the composite of empty structures

\[
1 \xrightarrow{\lambda^{-1}} 1 \otimes 1 \xrightarrow{\varphi_1 \otimes \varphi_1} F\emptyset \otimes F\emptyset \xrightarrow{F\emptyset \otimes F\emptyset} FN \otimes FM \xrightarrow{\varphi_N \otimes \varphi_M} F(N + M) \xrightarrow{F[iN \oplus iM]} F(N +_Y M)
\]

via \(F\emptyset \cong F(\emptyset + \emptyset)\).

Inspired by this embedding, we view \(\text{FCosp}an\) as a dagger compact category, and this embedding as a dagger monoidal functor. To be more precise, we define the monoidal product of objects \(X\) and \(Y\) of \(\text{FCosp}an\) to be their disjoint union \(X + Y\), and define the monoidal product of decorated cospan \((X \xrightarrow{i, iX} N \xleftarrow{o, oY} Y, 1 \xrightarrow{s} FN)\) and \((X' \xrightarrow{i', i'X'} N' \xleftarrow{o', o'Y'} Y', 1 \xrightarrow{s'} FN')\) to be

\[
\begin{pmatrix}
N + N' & F(N + N') \\
X + X' & Y + Y'
\end{pmatrix}
\]

\[
\xrightarrow{
\begin{pmatrix}
i_X + i_{X'} & o_Y + o_{Y'} \\
i_{X'} & o_Y + o_{Y'}
\end{pmatrix}
\}
\]

\[
\xleftarrow{
\begin{pmatrix}
\varphi_{N,N'} & (s \otimes t) \circ \lambda^{-1}
\end{pmatrix}
\}
\]

The functoriality of this monoidal product, as far as the structure elements are concerned, follows from the fact the isomorphism of pushouts

\[
(N + N') +_Y (M + M') \cong (N +_Y M) + (N'_Y, M').
\]

The functoriality for the cospan part follows from the properties of the coproduct, as it does for the monoidality of the embedding of \(\text{FinSet}\) into \(\text{Cospan}(\text{FinSet})\).

The dagger functor mimics exactly that on \(\text{Cospan}(\text{FinSet})\); we define the dagger to reflect the cospan part of a decorated cospan, keeping the same structure element:

\[
\dagger(X \xrightarrow{i} N \xleftarrow{o} Y, 1 \xrightarrow{s} FN) = (Y \xrightarrow{o} N \xleftarrow{i} X, 1 \xrightarrow{s} FN).
\]

The commutativity of the required coherence diagrams is then an immediate consequence of Lemma 4.3, and we arrive at the following proposition.

Proposition 4.4. Let \(F\) be a lax monoidal functor. Then, given the above structure, the category \(\text{FCosp}an\) is a dagger compact category.

We also write + for the monoidal product in \(\text{FCosp}an\).

4.4 Functors between decorated cospan categories

Decorated cospan allow us to understand the diagrammatic nature of structures on finite sets, such as circuit diagrams. Equally crucial to our understanding of circuit diagrams, however, is their semantics as discussed in the previous section, say via their interpretation as Dirichlet forms. Decorated cospan categories also give us the tools to understand these interpretations through dagger compact categories. In this section we show that this process of interpretation of a circuit diagram as a Dirichlet form—that is, a monoidal natural transformation between the lax monoidal functors defining these structures—preserves this dagger compact structure.
Lemma 4.5. Let 
\[(F, \phi), (G, \gamma) : \text{(FinSet,)} + \rightarrow (C, \otimes)\]
be lax monoidal functors, and let \(\theta : (F, \phi) \Rightarrow (G, \gamma)\) be a monoidal natural transformation between them. Then we may define a functor, moreover a strict monoidal dagger functor, 
\[T : FCospans \rightarrow GCospans\]
between the corresponding decorated cospan categories by letting finite sets \(X\) in \(FCospans\) map to the same finite set as an object of \(GCospans\), and letting morphisms 
\[(X \rightarrowtail N \leftarrowtail Y, 1 \xrightarrow{s} FN)\]
map to that with the element precomposed with the natural transformation \(\theta_N\) on the apex \(N\)
\[(X \rightarrowtail N \leftarrowtail Y, 1 \xrightarrow{\theta_N \circ s} GN).\]

Proof. We check that this proposed functor preserves identities, composition, monoidal composition, and dagger functors.

Identities: Let 
\[(X \xrightarrow{1_X} X \xleftarrow{1_X} X, 1 \xrightarrow{F! \circ \varphi_1} FX)\]
be the identity morphism on some object \(X\) in the category of \(F\)-decorated cospans. Now this morphism has \(T\)-image 
\[(X \xrightarrow{1_X} X \xleftarrow{1_X} X, 1 \xrightarrow{\theta_X \circ F! \circ \varphi_1} GX).\]
But we have the following diagram
\[
\begin{array}{ccc}
1 & \xrightarrow{\varphi_1} & F \otimes F \xrightarrow{F!} FX \\
\downarrow^{(1)} & & \downarrow^{\theta} \\
G \otimes G \xrightarrow{G!} GX & \xrightarrow{\theta_N} & GX
\end{array}
\]
where (1) commutes by the unitality condition for the monoidal functor \(\theta\), and (2) commutes by the naturality of \(\theta\). Thus we have the equality of elements \(\theta_X \circ F! \circ \varphi_1 = G! \circ \gamma_1 : 1 \mapsto GX\), and so \(T\) sends identity morphisms to identity morphisms.

Composition: Let 
\[(X \rightarrowtail N \leftarrowtail Y, 1 \xrightarrow{s} FN),\]
\[(Y \rightarrowtail M \leftarrowtail Z, 1 \xrightarrow{t} FM),\]
be morphisms in \(FCospans\). As the composition of the cospan part is by pushout in \text{Set} in both cases, and as \(T\) acts as the identity on these cospans, it is clear that \(T\) preserves composition of cospans. To see that \(T\) preserves composition of the structure elements, observe that the composite
of \( s \) and \( t \) is given by the top line in the following diagram, while the composite of their images under \( T \) is given by the bottom line:

\[
\begin{align*}
1 \xrightarrow{\lambda^{-1}} 1 \otimes 1 & \quad (1) \quad \theta_N \otimes \theta_M \quad (2) \quad \theta_{N+M} \quad (3) \quad \theta_{N+Y+M} \\
FN \otimes FM & \xrightarrow{\varphi_{N,M}} F(N + M) \xrightarrow{F[j_{N+M}]} F(N + Y + M) \\
GN \otimes GM & \xrightarrow{\gamma_{N,M}} G(N + M) \xrightarrow{G[j_{N+M}]} G(N + Y + M)
\end{align*}
\]

As (1) commutes by the interchange law, (2) by monoidality of the natural transformation \( \theta \), and (3) by its naturality, we see that the composite of the \( T \)-images of \( s \) and \( t \) is equal to the \( T \)-image of their composite, and so that \( T \) preserves composition.

**Monoidal product:** To show \( T \) is strict monoidal, it is enough to observe that for any pair of \( F \)-decorated cospans \( \hat{s} = (X \xrightarrow{t_N} N \xleftarrow{t_Y} Y, \ 1 \xrightarrow{s} FN) \) and \( \hat{t} = (X' \xrightarrow{t'_N} N' \xleftarrow{t'_Y} Y', \ 1 \xrightarrow{t} FN') \), we have the equality of \( G \)-decorated cospans

\[
T\hat{s} + T\hat{t} = T(\hat{s} + \hat{t})
\]

This is trivially true for the cospan part, and true for structure elements by the monoidality of \( \theta \).

**Dagger:** The dagger functor in both \( FCospans \) and \( GCospans \) simply reflects the cospan part of each morphism. As \( T \) acts as the identity on each such part, we trivially have that \( T \circ \dagger = \dagger \circ T \), as required.

5 The category of circuit diagrams

In Part I we defined a circuit to be a labelled graph with marked input and output terminals, as in the example:

![Circuit Diagram Example](https://example.com/circuit_diagram.png)

We now put our newfound understanding of decorated cospans into immediate use to formalise these notions of ‘marking terminals’ and composition of circuits. This gives a dagger compact category of circuit diagrams \( \text{Circ} \), with the dagger compact structure expressing standard operations on circuit diagrams.

We give two constructions of this category, first arriving at a category of cospans decorated by labelled graphs, and then showing this is the a full subcategory of the decategorification of the bicategory of cospans of labelled graphs.

In the following let \( \mathbb{F} \) be a field, and \( \mathbb{F}^+ \) be a subrig of positive elements.\(^5\) These stand in place in particular for our two motivating examples: the field \( \mathbb{R} \) of real numbers with subrig \((0, \infty)\) of positive numbers, modelling circuits of resistors; and the field \( \mathbb{R}(s) \) of real rational functions with subrig \( \mathbb{R}(s)^+ \) of real rational functions positive on the positive real axis, modelling passive linear circuits.

\(^5\)Recall that a rig is a ‘ring without negatives’
5.1 A decorated cospan construction

Observe that we may view a circuit over $\mathbb{F}^+$ as a cospan $(X \to N \leftarrow Y)$ in FinSet decorated with a $\mathbb{F}^+$-graph with vertex set equal to the apex $N$. This suggests a decorated cospan category. Indeed, we show that the map taking a finite set $N$ to the set of $\mathbb{F}^+$-graphs with vertex set $N$ in fact forms a lax monoidal functor. This allows us to apply Lemma 4.2 to construct a category of circuits.

To this end, define the functor

$$\text{Circuit} : (\text{FinSet}, +) \rightarrow (\text{Set}, \times)$$

to take a finite set $N$, as an object of FinSet, to the set Circuit$(N)$ of $\mathbb{F}^+$-graphs $(N, E, s, t, r)$ with vertex set $N$. On morphisms let it take a function $f : N \to M$ to the function that pushes labelled graph structures on a set $N$ forward onto the set $M$:

$$\text{Circuit}(f) : \text{Circuit}(N) \rightarrow \text{Circuit}(M);$$

$$(N, E, s, t, r) \mapsto (M, E, f \circ s, f \circ t, r).$$

Note that as this map simply acts by post-composition, our map Circuit is indeed functorial.

We then arrive at a lax monoidal functor by equipping this functor with the natural transformation

$$\rho_{N,M} : \text{Circuit}(N) \times \text{Circuit}(M) \rightarrow \text{Circuit}(N + M);$$

$$((N, E, s, t, r), (M, F, s', t', r')) \mapsto (N + M, E + F, s + s', t + t', [r, r']),$$

together with the unit map

$$\rho_1 : 1 \rightarrow \text{Circuit}(\emptyset);$$

$$\bullet \mapsto (\emptyset, \emptyset, \emptyset, \emptyset, \emptyset),$$

where we use $\emptyset$ to denote both the empty set and the unique function of the appropriate codomain with domain the empty set. The naturality of this collection of morphisms, as well as the coherence laws for lax monoidal functors, follow from the universal property of the coproduct.

Definition 5.1. We define

$$\text{Circ} = \text{CircuitCospan}$$

Proposition 4.4 then specialises to:

Corollary 5.2. The category Circ is a dagger compact category.

The different structures of this category capture different operations that can be performed with circuits. The composition expresses the fact that we can connect the outputs of one circuit to the inputs of the next, while the monoidal composition models the placement of circuits side-by-side. The symmetric monoidal structure allows us reorder input and output wires, and the compactness captures the interchangeability between input and output terminals of circuits—that is, the fact that we can choose any input terminal to our circuit and consider it instead as an output terminal, and vice versa, while the dagger functor expresses the fact that we may reverse the orientation of entire circuit components.

5.2 A bicategory of circuits

As mentioned above, the category Circ may also be viewed as a full subcategory of the bicategory of cospans of $\mathbb{F}^+$-labelled graphs. The construction of such a bicategory of cospans follows that of
Bénabou [8], although an alternate construction with the same property, using spans of cospans rather than maps of cospans as 2-morphisms, can be developed from the work of Rebro [43].

Recall that we define a graph to be a pair of functions \( s, t : E \to N \) where \( E \) and \( N \) are finite sets, and an \( L \)-graph to be a graph further equipped with a function \( r : E \to L \).

\[
\begin{array}{c}
L \\
\downarrow \epsilon \\
E
\end{array} \quad \begin{array}{c}
E \\
\downarrow \epsilon \\
E'
\end{array} \quad \begin{array}{c}
N \\
\downarrow v \\
N'
\end{array}
\]

Given \( L \)-graphs \( \Gamma = (E, N, s, t, r) \) and \( \Gamma' = (E', N', s', t', r') \), a morphism of \( L \)-graphs \( \Gamma \to \Gamma' \) is a pair of functions \( \epsilon : E \to E' \), \( v : N \to N' \) such that the following diagrams commute:

\[
\begin{array}{c}
L \\
\downarrow \epsilon \\
E
\end{array} \quad \begin{array}{c}
E \\
\downarrow \epsilon \\
E'
\end{array} \quad \begin{array}{c}
N \\
\downarrow v \\
N'
\end{array}
\]

These \( L \)-graphs and their morphisms form the category \( L{-}\text{Graph} \). Using results about colimits in the category of sets, it is straightforward to check that this category has finite colimits.

Then \( L \)-labelled graphs, cospans in \( L{-}\text{Graph} \), and maps of cospans form a bicategory \( \text{Cospan}(L{-}\text{Graph}) \) [8]. Moreover, this bicategory is compact closed [54]. Note that this requires choosing pushouts for each pushout square in \( L{-}\text{Graph} \). We make the following definition.

**Definition 5.3.** The bicategory \( 2{-}\text{Circ} \) is the full subbicategory of \( \text{Cospan}(\mathbb{F}^+{-}\text{Graph}) \) with objects those \( F^+{-}\text{graphs} \) with no edges.

It can be shown that every object in \( \text{Cospan}(\mathbb{F}^+{-}\text{Graph}) \) is self-dual, and this implies that \( 2{-}\text{Circ} \) is again a compact closed bicategory [54]. Note that the 1-categorical decategorification of this bicategory has objects finite sets and morphisms isomorphism classes of cospans in \( \mathbb{F}^+{-}\text{Graph} \) with feet such objects. Note that the notion of isomorphism for cospans of labelled graph is quite restrictive, amounting to no more than a renaming of the nodes and edges, so we may be lazy with our distinction between the notions of morphism of this category and of cospans themselves. As composition in this category is given by pushout, we see that this decategorification of \( 2{-}\text{Circ} \) is in fact isomorphic to \( \text{Circ} \).

## 6 Circuits as Lagrangian relations

In the first part of this paper, we explored the semantic content contained in circuit diagrams, leading to an understanding of circuit diagrams as expressing some relationship between the potentials and currents that can simultaneously be imposed on some subset, the so-called terminals, of the nodes of the circuit. We called this collection of possible relationships the behaviour of the circuit. While in that setting we used the concept of Dirichlet forms to describe this relationship, we saw in the end that describing circuits as Dirichlet forms does not allow for a straightforward notion of composition of circuits.

In this section, inspired by the principle of least action of classical mechanics in analogy with the principle of minimum power, we develop a setting for describing behaviours that allows for easy discussion of composite behaviours: Lagrangian subspaces of symplectic vector spaces. These Lagrangian subspaces provide a more direct, invariant perspective, comprising precisely the set of vectors describing the possible simultaneous potential and current readings at all terminals of a given circuit. As we shall see, one immediate and important advantage of this setting is that we may model wires of zero resistance.

Recall that we write \( \mathbb{F} \) for some field, with for the applications in mind this field usually the field \( \mathbb{R} \) of real numbers or \( \mathbb{R}(s) \) of real rational functions.
6.1 Symplectic vector spaces

A circuit made up of wires of positive resistance defines a function from boundary potentials to boundary currents. A wire of zero resistance, however, does not define a function: the principle of minimum power is obeyed as long as the potentials at the two ends of the wire are equal. More generally, we may thus think of circuits as specifying a set of allowed voltage-current pairs, or as a relation between boundary potentials and boundary currents. This set forms what is called a Lagrangian subspace, and is given by the graph of the differential of the power functional. More generally, Lagrangian submanifolds graph derivatives of smooth functions: they describe the point evaluated and the tangent to that point within the same space.

The material in this section is all known, and follows without great difficulty from the definitions. To keep this section brief we omit proofs. See any introduction to symplectic spaces, such as Cimasoni and Turaev [15] or Piccione and Tausk [42] for details.

Definition 6.1. Given a vector space $V$ over a field $\mathbb{F}$, a symplectic form $\omega: V \times V \to \mathbb{F}$ on $V$ is a antisymmetric nondegenerate bilinear form. That is, a symplectic form $\omega$ is a function $V \times V \to \mathbb{F}$ that obeys

(i) bilinearity: for all $\lambda \in \mathbb{F}$ and all $u, v \in V$ we have $\omega(\lambda u, v) = \omega(u, \lambda v) = \lambda \omega(u, v)$;

(ii) total isotropy: for all $v \in V$ we have $\omega(v, v) = 0$; and

(iii) nondegeneracy: given $v \in V$, $\omega(u, v) = 0$ for all $u \in V$ if and only if $u = 0$.

A symplectic vector space $(V, \omega)$ is a vector space $V$ equipped with a symplectic form $\omega$.

Given symplectic vector spaces $(V_1, \omega_1), (V_2, \omega_2)$, a symplectic map is a linear map

$$f: (V_1, \omega_1) \to (V_2, \omega_2)$$

such that $\omega_2(f(u), f(v)) = \omega_1(u, v)$ for all $u, v \in V_1$. A symplectomorphism is a symplectic map that is also an isomorphism.

A symplectic basis for a symplectic vector space $(V, \omega)$ is a basis $\{p_1, \ldots, p_n, q_1, \ldots, q_n\}$ such that $\omega(p_i, p_j) = \omega(q_i, q_j) = 0$ for all $1 \leq i, j \leq n$, and $\omega(p_i, q_j) = \delta_{ij}$ for all $1 \leq i, j \leq n$, where $\delta_{ij}$ is the Dirac delta, equal to 1 when $i = j$, and 0 otherwise. A symplectomorphism maps symplectic bases to symplectic bases, and conversely: any map that takes a symplectic basis to another symplectic basis is a symplectomorphism.

Example 6.2 (The symplectic vector space generated by a finite set). Given a finite set $N$, we consider the vector space $\mathbb{F}^N \oplus (\mathbb{F}^N)^*$ a symplectic vector space $(\mathbb{F}^N \oplus (\mathbb{F}^N)^*, \omega)$, with symplectic form

$$\omega(\{\phi, i\}, \{\phi', i'\}) = \phi'(i) - \phi(i').$$

Let $\{\phi_n\}_{n \in N}$ be the basis of $\mathbb{F}^N$ consisting of the functions $N \to \mathbb{F}$ mapping $n$ to 1 and all other elements of $n$ to 0, and let $\{(i_n)_{n \in N} \subseteq (\mathbb{F}^N)^*\}$ be the dual basis. Then $\{(\phi_n, 0), (0, i_n)\}_{n \in N}$ forms a symplectic basis for $\mathbb{F}^N \oplus (\mathbb{F}^N)^*$.

There are two common ways we will build symplectic spaces from other symplectic spaces: conjugation and summation. Given a symplectic form $\omega$, we may define its conjugate symplectic form $\overline{\omega} = -\omega$, and write the conjugate symplectic space $(V, \overline{\omega})$ as $\overline{V}$. Given two symplectic vector spaces $(U, \nu), (V, \omega)$, we consider their direct sum $U \oplus V$ a symplectic vector space with the symplectic form $\nu + \omega$, and call this the sum of the two symplectic vector spaces. Note that this is not a product in the category of symplectic vector spaces and symplectic maps.

The symplectic form provides a notion of orthogonal subspace, and this is useful for studying the symplectic form. Given a subspace $S$ of $V$, we define its complement

$$S^\perp = \{v \in V \mid \omega(v, s) = 0 \text{ for all } s \in S\}.$$
Note that this construction obeys the following identities, where $S$ and $T$ are subspaces of $V$:

\[
\dim S + \dim S^\circ = \dim V \\
(S^\circ)^\circ = S \\
(S + T)^\circ = S^\circ \cap T^\circ \\
(S \cap T)^\circ = S^\circ + T^\circ.
\]

Given a symplectic vector space $V = U \oplus U^*$, the subspace $U$ has the property of being a maximal subspace such that the symplectic form restricts to the zero form on $U$. Subspaces with this property are known as Lagrangian subspaces, and may be realised as the image of $U$ under symplectic isometries $V \to V$. Lagrangian subspaces may also be viewed as the subspaces that correspond to the collection of (point, tangent vector) pairs for quadratic forms on $V$.

**Definition 6.3.** Let $S$ be a linear subspace of a symplectic vector space $(V, \omega)$. We say that $S$ is isotropic if $\omega|_{S \times S} = 0$, and that $S$ is coisotropic if $S^\circ$ is isotropic. A subspace is Lagrangian if it is both isotropic and coisotropic.

Lagrangian subspaces are also known as Lagrangian correspondences and canonical relations. Note that a subspace $S$ is isotropic if and only if $S \subseteq S^\circ$. This fact helps with the following characterisations of Lagrangian subspaces.

**Proposition 6.4.** Given a subspace $L \subset V$ of a symplectic vector space $(V, \omega)$, the following are equivalent:

(i) $L$ is Lagrangian.

(ii) $L$ is maximally isotropic.

(iii) $L$ is minimally coisotropic.

(iv) $L = L^\circ$.

(v) $L$ is isotropic and $\dim L = \frac{1}{2} \dim V$.

From this proposition it follows easily that the direct sum of two Lagrangian subspaces in Lagrangian in the sum of their ambient spaces. We also observe that an advantage of isotropy is that there is a good way to take a quotient of a symplectic vector space by an isotropic subspace—that is, there is a way to put a natural symplectic structure on the quotient space.

**Proposition 6.5.** Let $S$ be an isotropic subspace of a symplectic vector space $(V, \omega)$. Then $S^\circ / S$ is a symplectic vector space with symplectic form $\omega'(v + S, u + S) = \omega(v, u)$.

*Proof.* The function $\omega'$ is a well-defined due to the isotropy of $S$—by definition adding any pair $(s, s')$ of elements of $S$ to a pair $(v, u)$ of elements of $S^\circ$ does not change the value of $\omega(v + s, u + s')$. As $\omega$ is a symplectic form, this function is too. □

### 6.2 Lagrangian subspaces from quadratic forms

Lagrangian subspaces are of relevance to us here as the behaviour of any passive linear circuit forms a Lagrangian subspace of the symplectic vector space generated by the nodes of the circuit. We think of this vector space as comprising two parts, a space $\mathbb{F}^N$ of potentials at each node, and a dual space $(\mathbb{F}^N)^*$ of currents. To make clear how circuits can be interpreted as Lagrangian subspaces, here we describe how Dirichlet forms on a finite set $N$ give rise to Lagrangian subspaces of $\mathbb{F}^N \oplus (\mathbb{F}^N)^*$. More generally, we show that there is a one-to-one correspondence between Lagrangian subspaces and quadratic forms.
Proposition 6.6. Let $N$ be a finite set. Given a quadratic form $Q$ over $\mathbb{F}$ on $N$, the subspace

$$L_Q = \{ (\phi, dQ_\phi) \mid \phi \in \mathbb{F}^N \} \subseteq \mathbb{F}^N \oplus (\mathbb{F}^N)^*,$$

where $dQ_\phi \in (\mathbb{F}^N)^*$ is the formal differential of $Q$ at $\phi \in \mathbb{F}^N$, is Lagrangian. Moreover, this construction gives a one-to-one correspondence

$$\left\{ \text{Quadratic forms over } \mathbb{F} \text{ on } N \right\} \mapsto \left\{ \text{Lagrangian subspaces of } \mathbb{F}^N \oplus (\mathbb{F}^N)^* \text{ with trivial intersection} \right\}.$$

Proof. The symplectic structure on $\mathbb{F}^N \oplus (\mathbb{F}^N)^*$ and our notation for it is given in Example 6.2.

Note that for all $n, m \in N$ the corresponding basis elements

$$\frac{\partial^2 Q}{\partial \phi_n \partial \phi_m} = dQ_{\phi_n}(\phi_m) = dQ_{\phi_m}(\phi_n),$$

so $dQ_{\phi}(\psi) = dQ_{\psi}(\phi)$ for all $\phi, \psi \in \mathbb{F}^N$. Thus $L_Q$ is indeed Lagrangian: for all $\phi, \psi \in \mathbb{F}^N$

$$\omega((\phi, dQ_\phi), (\psi, dQ_\psi)) = dQ_{\psi}(\phi) - dQ_{\phi}(\psi) = 0.$$

Observe also that for all quadratic forms $Q$ we have $dQ_0 = 0$, so the only element of $L_Q$ of the form $(0, i)$, where $i \in (\mathbb{F}^N)^*$; is $(0, 0)$. Thus $L_Q$ has trivial intersection with the subspace $(0) \oplus (\mathbb{F}^N)^*$ of $\mathbb{F}^N \oplus (\mathbb{F}^N)^*$. This $L_Q$ construction forms the leftward direction of the above correspondence.

For the rightward direction, suppose that $L$ is a Lagrangian subspace of $\mathbb{F}^N \oplus (\mathbb{F}^N)^*$ such that $L \cap \{(0) \oplus (\mathbb{F}^N)^*\} = \{(0, 0)\}$. Then for each $\phi \in \mathbb{F}^N$, there exists a unique $i_\phi \in (\mathbb{F}^N)^*$ such that $\langle \phi, i_\phi \rangle \in L$. Indeed, if $i_\phi$ and $i_\phi'$ were distinct elements of $(\mathbb{F}^N)^*$ with this property, then by linearity $(0, i_\phi - i_\phi')$ would be a nonzero element of $L \cap \{(0) \oplus (\mathbb{F}^N)^*\}$, contradicting the hypothesis about trivial intersection. We thus can define a function, indeed a linear map, $\mathbb{F}^N \to (\mathbb{F}^N)^* : \phi \mapsto i_\phi$. This defines a bilinear form $Q(\phi, \psi) = i_\phi(\psi)$ on $\mathbb{F}^N \oplus \mathbb{F}^N$, and so $Q(\phi) = i_\phi(\phi)$ defines a quadratic form on $\mathbb{F}^N$.

Moreover, $L$ is Lagrangian, so

$$\omega((\phi, i_\phi), (\psi, i_\psi)) = i_\phi(\psi) - i_\phi(\psi) = 0,$$

and so $Q(-, -)$ is a symmetric bilinear form. This gives a one-to-one correspondence between Lagrangian subspaces of specified type, symmetric bilinear forms, and quadratic forms, and so in particular gives the claimed one-to-one correspondence.

In particular, every Dirichlet form defines a Lagrangian subspace. Note that the differential of a Dirichlet form is a doubly infinitesimally stochastic operator. For more details about such operators, see Baez and Pollard [6].

6.3 Composing Lagrangian subspaces

Recall that a relation between sets $X$ and $Y$ is a subset $R$ of their product $X \times Y$. Furthermore, given relations $R \subseteq X \times Y$ and $S \subseteq Y \times Z$, there is a composite relation $(S \circ R) \subseteq X \times Z$ given by pairs $(x, z)$ such that there exists $y \in Y$ with $(x, y) \in R$ and $(y, z) \in S$—a direct generalisation of function composition. A Lagrangian relation between symplectic vector spaces $V_1$ and $V_2$ is a relation between $V_1$ and $V_2$ that forms a Lagrangian subspace of the symplectic vector space $V_1 \oplus V_2$. This gives us a way to think of certain Lagrangian subspaces, such as those arising from circuits, as morphisms, giving a way to compose them.

Definition 6.7. A Lagrangian relation $L : V_1 \to V_2$ is a Lagrangian subspace $L$ of $V_1 \oplus V_2$. 

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This is a generalisation of the notion symplectomorphism: any symplectomorphism \( f : V_1 \to V_2 \) forms a Lagrangian subspace when viewed as a relation \( f \subseteq V_1 \oplus V_2 \). More generally, any symplectic map \( f : V_1 \to V_2 \) forms an isotropic subspace when viewed as a relation in \( V_1 \oplus V_2 \).

Importantly for us, the composite of two Lagrangian relations is again a Lagrangian relation.

**Proposition 6.8.** Let \( L : V_1 \to V_2 \) and \( L' : V_2 \to V_3 \) be Lagrangian relations. Then their composite relation \( L' \circ L \) is a Lagrangian relation \( V_1 \to V_3 \).

We prove this proposition by way of two lemmas detailing how the Lagrangian property is preserved under various operations. The first lemma says that the intersection of a Lagrangian space with a coisotropic space is in some sense Lagrangian, once we account for the complement.

**Lemma 6.9.** Let \( L \subseteq V \) be a Lagrangian subspace of a symplectic vector space \( V \), and \( S \subseteq V \) be an isotropic subspace of \( V \). Then \( (L \cap S^o) + S \subseteq V \) is Lagrangian in \( V \).

**Proof.** Recall from Proposition 6.4 that a subspace is Lagrangian if and only if it is equal to its complement. The lemma is then immediate from the way taking the symplectic complement interacts with sums and intersections:

\[
((L \cap S^o) + S)^o = (L \cap S^o)^o \cap S^o = (L^o + (S^o)^o) \cap S^o = (L + S) \cap S^o = (L \cap S^o) + (S \cap S^o) = (L \cap S^o) + S.
\]

Since \( (L \cap S^o) + S \) is equal to its complement, it is Lagrangian. \( \square \)

The second lemma says that if a subspace of a coisotropic space is Lagrangian, taking quotients by the complementary isotropic space does not affect this.

**Lemma 6.10.** Let \( L \subseteq V \) be a Lagrangian subspace of a symplectic vector space \( V \), and \( S \subseteq L \) be an isotropic subspace of \( V \) contained in \( L \). Then \( L/S \subseteq S^o/S \) is Lagrangian in the quotient symplectic space \( S^o/S \).

**Proof.** As \( L \) is isotropic and the symplectic form on \( S^o/S \) is given by \( \omega'(v + S, u + S) = \omega(v, u) \), the quotient \( L/S \) is immediately isotropic. Recall from Proposition 6.4 that an isotropic subspace \( S \) of a symplectic vector space \( V \) is Lagrangian if and only if \( \dim S = \frac{1}{2} \dim V \). Also recall that for any subspace \( \dim S + \dim S^o = \dim V \). Thus

\[
\dim(L/S) = \dim L - \dim S = \frac{1}{2} \dim V - \dim S
\]

\[
= \frac{1}{2} \left( \dim S + \dim S^o \right) - \dim S = \frac{1}{2} \left( \dim S^o - \dim S \right) = \frac{1}{2} \dim(S^o/S).
\]

Thus \( L/S \) is Lagrangian in \( S^o/S \). \( \square \)

Combining these two lemmas gives a proof that the composite of two Lagrangian relations is again a Lagrangian relation.

**Proof of Proposition 6.8.** Let \( \Delta \) be the diagonal subspace

\[
\Delta = \{(0, v_2, v_2, 0) \mid v_2 \in V_2\} \subseteq \overline{V_1} \oplus V_2 \oplus \overline{V_2} \oplus V_3.
\]

Observe that \( \Delta \) is isotropic, and has coisotropic complement

\[
\Delta^o = \{(v_1, v_2, v_2, v_3) \mid v_i \in V_i\} \subseteq \overline{V_1} \oplus V_2 \oplus \overline{V_2} \oplus V_3.
\]

As \( \Delta \) is the kernel of the restriction of the projection map \( \overline{V_1} \oplus V_2 \oplus \overline{V_2} \oplus V_3 \to \overline{V_1} \oplus V_3 \) to \( \Delta^o \), and after restriction this map is still surjective, the quotient space \( \Delta^o/\Delta \) is isomorphic to \( \overline{V_1} \oplus V_3 \).

Now, by definition of composition of relations,

\[
L' \circ L = \{(v_1, v_3) \mid \text{there exists } v_2 \in V_2 \text{ such that } (v_1, v_2) \in L, (v_2, v_3) \in L'\}.
\]
But note also that
\[ L \oplus L' = \{(v_1, v_2, v_3') \mid (v_1, v_2) \in L, (v_2', v_3) \in L'\}, \]
so
\[ (L \oplus L') \cap \Delta^\circ = \{(v_1, v_2, v_3) \mid \text{there exists } v_2 \in V_2 \text{ such that } (v_1, v_2) \in L, (v_2, v_3) \in L'\}. \]
Quotienting by \( \Delta \) then gives
\[ L' \circ L = ((L \oplus L') \cap \Delta^\circ) + \Delta)/\Delta. \]
As \( L' \oplus L \) is Lagrangian in \( \overline{V_1} \oplus V_2 \oplus \overline{V_2} \oplus V_3 \), Lemma 6.9 says that \( (L' \oplus L) / \Delta^\circ \Delta \) is also Lagrangian in \( \overline{V_1} \oplus V_2 \oplus \overline{V_2} \oplus V_3 \). Lemma 6.10 thus shows that \( L' \circ L \) is Lagrangian in \( \Delta^\circ/\Delta = \overline{V_1} \oplus V_3 \), as required.

Note that this composition is associative. We shall see later that this composition agrees with composition of Dirichlet forms, and hence also composition of circuits.

6.4 The dagger compact category of Lagrangian relations

Lagrangian relations solve the identity problems we had with Dirichlet forms: given a symplectic vector space \( V \), the Lagrangian relation id : \( V \to V \) specified by the Lagrangian subspace
\[ \text{id} = \{(v, v) \mid v \in V\} \subseteq \overline{V} \oplus V, \]
acts as an identity for composition of relations. We thus have a category.

**Definition 6.11** \((\text{LagrRel})\). We write \( \text{LagrRel} \) for the category with objects symplectic vector spaces and morphisms Lagrangian relations.

In fact the move to the setting of Lagrangian relations, rather than Dirichlet forms, adds far richer structure than just identity morphisms. The category \( \text{LagrRel} \) can be viewed as endowed with the structure of a dagger compact category in which every object is equipped with a commutative special dagger-Frobenius monoid. We lay this out in steps.

**Symmetric monoidal structure**

We define the monoidal product of two objects of \( \text{LagrRel} \) to be their direct sum, and similarly define the monoidal product of two morphisms to be the direct product—note that the direct product of two Lagrangian subspaces is again Lagrangian in the product of their ambient spaces, and the zero dimensional vector space \( \{0\} \) acts as an identity for this product. Indeed, defining for all objects \( U, V, W \) in \( \text{LagrRel} \) unitors
\[ \lambda_V = \{(0, v, v) \} \subseteq \overline{0} \oplus V \oplus V, \]
\[ \rho_V = \{(v, 0, v) \} \subseteq V \oplus \{0\} \oplus V, \]
associators
\[ \alpha_{U,V,W} = \{(u, v, w, u, v, w) \} \subseteq (U \oplus V) \oplus W \oplus U \oplus (V \oplus W), \]
and swaps
\[ \sigma_{U,V} = \{(u, v, u, v) \mid u \in U, v \in V\} \subseteq \overline{U} \oplus V \oplus V \oplus U, \]
we have a symmetric monoidal category. Indeed, note that all these structure maps come from symplectomorphisms between the domain and codomain. From this viewpoint it is immediate that all the necessary diagrams commute, and so we have a symmetric monoidal category.
Duals for objects
Each object $V$ of LagrRel is dual to its conjugate space $\mathcal{V}$, with cup $\eta : \{0\} \to \mathcal{V} \oplus V$ given by
$$\eta = \{(0,v,v) \mid v \in V\} \subseteq \{0\} \oplus \mathcal{V} \oplus V$$
and cap $\epsilon : V \oplus \mathcal{V} \to \{0\}$ given by
$$\epsilon = \{(v,v,0) \mid v \in V\} \subseteq V \oplus \mathcal{V} \oplus \{0\}.$$ 
It is straightforward to check these satisfy the snake equations.

Dagger functor
Given symplectic vector spaces $U, V$, observe that the map
$$\dagger : U \oplus V \to V \oplus U; \quad (u,v) \mapsto (v,u)$$
takes Lagrangian subspaces of the domain to Lagrangian subspaces of the codomain. Thus we can view it as a map $\dagger$ taking morphisms $L : U \to V$ of LagrRel to morphisms $\dagger(L) : V \to U$. This defines a functor which is the identity on objects, and in fact a symmetric monoidal dagger functor: by inspection the dagger of the monoidal product of maps is equal to the monoidal product of their daggers, and it takes each structural morphism to its inverse.

This dagger functor plays nicely with the compactness: it is clear that $\eta^\dagger = \epsilon \circ \sigma$. We thus have a dagger compact category.

The commutative special dagger-Frobenius structure on each object will be explored in the next section on corelations. Before discussing this, we first use the compact closed structure we have just defined to understand composition in this category from a new perspective, one that will be useful in understanding how this composition relates to composition of decorated cospans.

6.5 Names of Lagrangian relations
This brief subsection illustrates a guiding principle of this paper: duals for objects allow us to blur the distinction between standard categorical composition and monoidal composition. We will make use of this when we prove the functoriality of the black box functor.

Observe that a Lagrangian relation $L : \{0\} \to V$ is the same as a Lagrangian subspace of $V$. Moreover, given a Lagrangian relation $L : U \to V$,

![Diagram 1](image1.png)

compactness allows us to view it as a Lagrangian relation $\{0\} \to U \oplus V$:

![Diagram 2](image2.png)
We call this subspace the **name** of the Lagrangian relation $L$; indeed, we have fittingly used this one-to-one correspondence between morphisms and their names to define Lagrangian relations.

By compactness, we have the equation

$$L \circ U = L \circ W$$

Here the right hand side is the name of the composite $M \circ L$ of Lagrangian relations, while the left hand side is the sum of the names of $L$ and $M$ post-composed with the Lagrangian relation $U$.

Thus this relation above, the product of a cap and two identity maps, enacts composition of Lagrangian relations. The process outlined in the proof of Proposition 6.8, that the composite of two Lagrangian relations is again a Lagrangian relation, makes use of this fact. We shall also return to it when discussing our functor $\text{Circ} \to \text{LagrRel}$. Note the similarity in form between our diagrams of cospans and of names of Lagrangian relations.

### 7 Ideal wires and corelations

In the previous section our exploration of the meaning of circuit diagrams culminated with our understanding of behaviours as Lagrangian subspaces. We now turn our attention to how circuit components fit together, and the category of operations. In this section we shall see that the algebra of connections is described by the concept of corelations, a generalisation of the notion of function that forgets the directionality from the domain to the codomain. We then observe that Kirchhoff’s laws follow directly from interpreting these structures in the category of linear relations.

#### 7.1 Ideal wires

To motivate the definition of this category, let us start with a set of input terminals $X$, and a set of output terminals $Y$. We may connect these terminals with ideal wires of zero impedance, whichever way we like—input to input, output to output, input to output—producing something like:

In doing so, we introduce a notion of equivalence on our terminals, where two terminals are equivalent if we, or if electrons, can traverse from one to another via some sequence of wires. Because of this, we consider our perfectly-conducting components to be equivalence relations on $X + Y$, transforming
The dotted lines indicate equivalence classes of points, while for reference the grey lines indicate ideal wires connecting these points, running through a central hub.

Given another circuit of this sort, say from sets $Y$ to $Z$,

we may combine these circuits into a circuit $X$ to $Z$ by taking the transitive closure of the two equivalence relations, and then restricting this to an equivalence relation on $X + Z$. This in fact defines a dagger compact category of fundamental importance.

### 7.2 The category of corelations

In the category of sets we hold the fundamental relationship between sets to be that of functions. These encode the idea of a deterministic process that takes each element of one set to a unique element of the other. For the study of networks this is less appropriate, as the relationship between terminals is not an input-output one, but rather one of interconnection.

In particular, the direction of a function becomes irrelevant, and to describe these interconnections via the category of sets we must develop an understanding of how to compose functions head to head and tail to tail. We have so far used cospans and pushouts to address this. Cospans, however, come with an apex, which represents extraneous structure beyond the two sets we wish to specify a relationship between. Correlations are the result of the omission of this information.

**Definition 7.1.** A corelation $\alpha : X \rightarrow Y$ between sets $X$ and $Y$ is a surjective function $\alpha : X + Y \rightarrow A$, for some set $A$.

By the universal property of the coproduct, corelations $X + Y \rightarrow A$ are in one-to-one correspondence with jointly-epic cospans $X \rightarrow A \leftarrow Y$. Given corelations $\alpha : X \rightarrow Y$, $\beta : Y \rightarrow Z$, we may thus compose them as cospans, before restricting the apex to the joint-image of the feet, to obtain a composite corelation $\beta \circ \alpha : X + Z \rightarrow C$.

Ellerman gives a detailed treatment of this concept from a logic viewpoint in [18], while basic category theoretic aspects can be found in Lawvere and Rosebrugh [31].

We may equivalently view a corelation as a partition of $X + Y$. That is, for finite sets $X$ and $Y$, a corelation is a collection of nonempty subsets $\alpha = \{A_1, A_2, \ldots, A_n\}$ of $X + Y$ such that
(i) $\alpha$ does not contain the empty set.

(ii) $\bigcup_{i=1}^{n} A_i = X + Y$.

(iii) $A_i \cap A_j = \emptyset$ whenever $i \neq j$.

Composition of corelations is then given by finding the finest partition on $X + Y + Z$ that is coarser than both $\alpha$ and $\beta$ when restricted to $X + Y$ and $Y + Z$ respectively, and then restricting this to a partition on $X + Z$. More explicitly, the corelation $\beta \circ \alpha = \{C_1, C_2, \ldots, C_m\}$ is the unique set of pairwise disjoint $C_i$ of the form

$$C_i = \bigcup_j A_j \cap X \cup \bigcup_k B_k \cap Z$$

for $j, k$ varying over indices such that

$$\bigcup_j A_j \cap Y = \bigcup_k B_k \cap Y.$$ 

This rule is associative as both pairwise methods of composing relations $\alpha : X \to Y$, $\beta : Y \to Z$, and $\gamma : Z \to W$ amount to finding the finest partition on $X + Y + Z + W$ that is coarser than each of $\alpha$, $\beta$, and $\gamma$ when restricted to the relevant subset, and then restricting this partition to a partition on $X + W$; reference to the motivating spider diagrams makes this clear.

The identity corelation $1_X : X \to X$ on a set $X$ is the map $[1_X, 1_X] : X + X \to X$. Equivalently, it is the partition of $X + X$ such that each partition comprises exactly two elements: an element $x \in X$ considered as an element of both the first and then the second summand of $X + X$. We thus define a category.

**Definition 7.2.** Let $\text{Corel}$ be the category with objects finite sets and morphisms corelations between finite sets.

This category is exactly what the motivating diagrams suggest. More precisely, it is the PROP for extraspecial Frobenius monoids. The details for this can be assembled from Baez–Erbele [4] and Lack [30]. The name arises as this construction is dual to the more well-known notion of relation. Note that neither is a generalisation of the other; the key property of corelations here is that it forms a compact category with monoidal product disjoint union of sets. This is not true of the category of relations.

Indeed, considering Corel as a monoidal category in this way, and with a dagger functor given by simply considering a map $X + Y \to A$ as a map $Y + X \to A$, we have the following relationship with $\text{Cospan(FinSet)}$.

**Proposition 7.3.** We have a pair of strict monoidal dagger functors

$$\begin{array}{ccc}
\text{Cospan(FinSet)} & \xrightarrow{\text{corelations}} & \text{Corel}
\end{array}$$

that compose to the identity functor on Corel.

**Proof.** These functors map finite sets to themselves. We thus need only discuss how they act on morphisms.

A cospan in $\text{FinSet}$ comprises a pair of functions $X \xrightarrow{f} N \xleftarrow{g} Y$. Restricting the apex $N$ down to the joint image $f(X) \cup g(Y)$ gives a jointly-epic cospan $X \xrightarrow{f} f(X) \cup g(Y) \xleftarrow{g} Y$, and so a corelation $X \to Y$. As the elements of the apex not in the image of maps from the feet play only a trivial role in the pushout, and hence in composition of cospans, this map is functorial.

For the reverse functor, we simply consider any corelation $X \to Y$ a jointly-epic cospan $X \to A \leftarrow Y$. It is immediate that these functors have the properties stated in the proposition. \qed
Via the rightward functor in the above proposition, the wide monoidal dagger embedding of FinSet into Cospan(FinSet) gives rise to a wide monoidal dagger embedding FinSet \rightarrow Corel. Moreover, composing with the dagger functor, we also have a wide monoidal dagger embedding FinSet^{op} \rightarrow Corel. Corelations give a method of composing functions independent of the direction of the function. Given some not necessarily directed path of functions, for example

\[ A \rightarrow B \leftarrow C \leftarrow D \rightarrow E \rightarrow F, \]

considering these functions as corelations gives a way to compose them.

In particular, while this mode of composition is simply composition of functions for two functions head to tail, and turning a cospan into a corelation for two functions head to head, for two tail to tail functions \( C \leftarrow D \rightarrow E \) we compute the composite of cospans

\[
\begin{array}{ccc}
C & \xleftarrow{1_C} & \downarrow & \xrightarrow{1_E} & E \\
\downarrow & & & & \downarrow \\
C & \xleftarrow{1_C} & \downarrow & \xrightarrow{1_E} & E
\end{array}
\]

before restricting the apex to arrive at a surjective function from \( C + E \). This implies the following lemma.

**Lemma 7.4.** Let

\[
\begin{array}{ccc}
P & \xleftarrow{P} & \downarrow & \xrightarrow{P} & B \\
A & \downarrow & \downarrow & \downarrow & \downarrow \\
N & \xleftarrow{N} & \downarrow & \xrightarrow{N} & B
\end{array}
\]

be a pushout square in FinSet. Then the composites of corelations \( A \rightarrow P \leftarrow B \) and \( A \leftarrow N \rightarrow B \) are equal.

### 7.3 Potentials on corelations

Chasing our interpretation of corelations as ideal wires, our aim for the remainder of this section is to build a functor

\[ S : Corel \rightarrow \text{LagrRel} \]

that expresses this interpretation. We break this functor down into the product of two parts, according to the behaviours of potentials and currents respectively.

The consideration of potentials gives a functor

\[ \Phi : Corel \rightarrow \text{LinRel}, \]

where LinRel is the category of \( F \)-vector spaces and linear relations. This functor is a generalisation of the contravariant functor \( \text{FinSet} \rightarrow \text{Vect} \) that maps a set to the vector space of \( F \)-valued functions on that set.

To this end, on the objects, finite sets \( X \), define that \( \Phi(X) \) be the vector space \( F^X \). On the morphisms, corelations \( \alpha : X \rightarrow Y \), define \( \Phi(\alpha) \) to be the linear subspace of \( F^X \oplus F^Y \) comprising functions \( \phi = [\phi_X, \phi_Y] : X + Y \rightarrow F \) that are constant on elements of \( \alpha \). Note that \( \Phi(\alpha) \) thus has dimension \( \#\alpha \). Again it is straightforward to check that the image under this map of the identity corelation on \( X \) is the identity relation on \( F^X \).

We check this map \( \Phi \) preserves composition. Let \( \alpha : X \rightarrow Y \) and \( \beta : Y \rightarrow Z \) be corelations. As \( \Phi \) maps corelations to relations, it is enough to check both inclusions \( \Phi(\beta) \circ \Phi(\alpha) \subseteq \Phi(\beta \circ \alpha) \) and \( \Phi(\beta \circ \alpha) \subseteq \Phi(\beta) \circ \Phi(\alpha) \).
Next we consider the case of currents, described by a functor potentials on corelations interpreted as ideal wires. We now do the same for currents. 

\[ \phi(c) = \phi_{X+Y+Z}(c_0) = \phi_{X+Y+Z}(c_1) = \cdots = \phi_{X+Y+Z}(c_n) = \phi(c') \]

as required.

### 7.4 Currents on corelations

Next we consider the case of currents, described by a functor

\[ I : \text{Corel} \rightarrow \text{LinRel}. \]

This functor \( I \) is a generalisation of the covariant functor \( \text{Set} \rightarrow \text{Vect} \) that maps a set to the vector space of \( \mathbb{F} \)-linear combinations of elements of that set.

Indeed, on objects define, for finite sets \( X \), that \( I(X) \) be the vector space \( (\mathbb{F}^X)^* \). Moreover, given a corelation \( \alpha : X \rightarrow Y \), define \( I(\alpha) \) to be the linear relation comprising precisely those \( (i_X, i_Y) \in (\mathbb{F}^X)^* \oplus (\mathbb{F}^Y)^* \) such that for all \( A_i \in \alpha \) the sum of the coefficients of the elements of \( A_i \cap X \) is equal to that for \( A_i \cap Y \):

\[ \sum_{x \in A_i \cap X} \lambda_x = \sum_{y \in A_i \cap Y} \lambda_y. \]

Note that this is indeed a linear subspace, and in fact one of dimension \( \#(X+Y) - \# \alpha \). It is easily checked that the image of the identity corelation is the identity relation.

We check this map \( I \) preserves composition. Again it is enough to check inclusions \( I(\beta) \circ I(\alpha) \subseteq I(\beta \circ \alpha) \) and \( I(\beta \circ \alpha) \subseteq I(\beta) \circ I(\alpha) \).
Thus these constraints define an affine subspace of \( \mathbb{F} \). Let \((i_X, i_Z) \in I(\beta) \circ I(\alpha)\), with \(i_X = \sum_{x \in X} \lambda_x x\) and \(i_Z = \sum_{z \in Z} \lambda_z z\). Note that this implies that there is some \(i_Y = \sum_{y \in Y} \lambda_y y\) such that \((i_X, i_Y) \in I(\alpha)\), \((i_Y, i_Z) \in I(\beta)\). Then for each \(C_i \in \beta \circ \alpha\) we have

\[
\sum_{x \in C_i \cap X} \lambda_x = \sum_{x \in A_j \cap X} \lambda_x = \sum_{y \in A_j \cap Y} \lambda_y = \sum_{z \in B_k \cap Z} \lambda_z = \sum_{z \in C_i \cap Z} \lambda_z.
\]

(By definition of \(I(\alpha)\))

(See composition of corelations)

(By definition of \(I(\beta)\))

Thus \((i_X, i_Z) \in I(\beta \circ \alpha)\).

\(I(\beta \circ \alpha) \subseteq I(\beta) \circ I(\alpha)\): The reverse inclusion requires a bit more effort. Let \((i_X, i_Z) \in I(\beta \circ \alpha)\). We wish to construct some \(i_Y = \sum_{y \in Y} \lambda_y y\) such that \((i_X, i_Y) \in I(\alpha)\) and \((i_Y, i_Z) \in I(\beta)\). This means that we must find a vector \(i_Y \in (\mathbb{F}^Y)^*\) satisfying the \#\(\alpha\) linear constraints

\[
\sum_{y \in A_j \cap Y} \lambda_y = \sum_{x \in A_j \cap X} \lambda_x
\]

and the \#\(\beta\) linear constraints

\[
\sum_{y \in B_k \cap Y} \lambda_y = \sum_{z \in B_k \cap Z} \lambda_z.
\]

Note, however, that for each \(C_i \in \beta \circ \alpha\), summing over the \(A_j\) and \(B_k\) that intersect \(C_i\) a linear dependence between these constraints themselves, as

\[
\sum_{y \in A_j \cap Y} \lambda_y = \sum_{x \in C_i \cap X} \lambda_x = \sum_{z \in C_i \cap Z} \lambda_z = \sum_{y \in B_k \cap Y} \lambda_y.
\]

Moreover, as each element of \(Y\) lies in exactly one element of each \(\alpha\) and \(\beta\), we may view them as edges in a graph on \(\alpha + \beta\), with exactly \#(\(\beta \circ \alpha\)) connected components. This implies that

\[
\#Y > \#\alpha + \#\beta - \#(\beta \circ \alpha).
\]

Thus these constraints define an affine subspace of \((\mathbb{F}^Y)^*\) of positive dimension, and hence we can always find a vector \(i_Y\) with the desired property. This proves the claim. Using elementary methods, an algorithm can also be given to construct an explicit solution.

### 7.5 The functor from Corel to LagrRel

We have now defined functors that, when interpreting corelations as connections of ideal wires, describe the behaviours of the currents and potentials at the terminals of these wires. In this section, we combine these to define a single functor

\[
S : Corel \rightarrow LagrRel
\]
describing the behaviour of both currents and potentials as a Lagrangian subspace.

Let $X$ be a finite set. We define

$$S(X) = \mathbb{F}^X \oplus (\mathbb{F}^X)^*.$$

Let $\alpha : X \to Y$ be a corelation. We define

$$S(\alpha) = I(\alpha) \oplus \Phi(\alpha) \subseteq \mathbb{F}^X \oplus (\mathbb{F}^X)^* \oplus \mathbb{F}^Y \oplus (\mathbb{F}^Y)^*.$$

To check this is well-defined we need only check $S(\alpha)$ is Lagrangian. This is true: it is isotropic as

$$\omega((i_X, i_Y, \phi_X, \phi_Y), (i_X', i_Y', \phi_X', \phi_Y')) = \phi_X'(i_X) - \phi_X(i_X') + \phi_Y(i_Y') - \phi_Y'(i_Y)$$

$$= \phi_X' \left( \sum_{x \in X} \lambda_x x \right) - \phi_X \left( \sum_{x \in X} \lambda_x x \right) + \phi_Y \left( \sum_{y \in Y} \lambda_y y \right) - \phi_Y' \left( \sum_{y \in Y} \lambda_y y \right)$$

$$= \sum_{x \in X} \lambda_x \phi_X'(x) - \sum_{x \in X} \lambda_x \phi_X(x) + \sum_{y \in Y} \lambda_y \phi_Y'(y) - \sum_{y \in Y} \lambda_y \phi_Y(y)$$

$$= \sum_{x \in X} \lambda_x \phi_X'(x) - \sum_{y \in Y} \lambda_y \phi_Y'(y) + \sum_{x \in X} \lambda_x \phi_X(x) - \sum_{y \in Y} \lambda_y \phi_Y(y)$$

$$= \sum_{A_J \in \alpha} \left( \sum_{x \in A_J \cap X} \lambda_x - \sum_{y \in A_J \cap Y} \lambda_y \right) k_{A_J} + \sum_{A_J \in \alpha} \left( \sum_{x \in A_J \cap X} \lambda_x - \sum_{y \in A_J \cap Y} \lambda_y \right) k_{A_J}$$

$$= 0$$

and has dimension equal to $\#(X + Y) - \#\alpha + \#\alpha = \#(X + Y)$.

We have thus shown we do indeed have a functor $S : \text{Corel} \to \text{LagrRel}$. In the next section we shall see that this functor provides the engine of our black box function, playing the key role in showing that we may indeed treat circuit components as black boxes: that is, that circuits that behave the same compose the same.

**Remark 7.5.** We make a remark on our conventions for string diagrams representing the cups and caps of the dualities in Corel and LagrRel.

Note that by the cap duality diagram

$$\begin{array}{c}
X \\
\cap \\
\cap \\
X
\end{array}$$

in Corel we mean the corelation $\cup_X : (X + X \overset{[1, 1]}{\rightarrow} X \leftarrow \emptyset)$, whereas by the cap

$$\begin{array}{c}
& \\
\cap \\
\cap \\
&
\end{array}$$

in LagrRel we mean the Lagrangian relation $\cup_V : V \oplus V \rightarrow 0$ given by the Lagrangian subspace $\{(v, v) \in V \oplus V \mid v \in V\}$. Although we represent them similarly, the functor $S$ does not map these directly onto each other: the image of $\cup_X$ under $S$ is the Lagrangian relation $S(\cup_X) : \mathbb{F}^X \oplus (\mathbb{F}^X)^* \oplus \mathbb{F}^X \oplus (\mathbb{F}^X)^* \rightarrow 0$ given by the Lagrangian subspace

$$\{(\phi, i, \phi, -i) \mid \phi \in \mathbb{F}^X, i \in (\mathbb{F}^X)^* \} \subseteq \mathbb{F}^X \oplus (\mathbb{F}^X)^* \oplus \mathbb{F}^X \oplus (\mathbb{F}^X)^*,$$
and in particular a Lagrangian relation $\mathbb{F}^X \oplus (\mathbb{F}^X)^* \oplus \mathbb{F}^X \oplus (\mathbb{F}^X)^* \to 0$ and not $\mathbb{F}^X \oplus (\mathbb{F}^X)^* \oplus \mathbb{F}^X \oplus (\mathbb{F}^X)^* \to 0$. As the cup diagrams in these categories simply denote the dagger image of these caps, the analogous statements apply to them too.

This is an expression of the fact that in cospan categories and Corel there is a canonical self-duality, while for Lagrangian relations a basis must be picked before there is an isomorphism between a symplectic vector space and its conjugate, the conjugate being a canonical dual object. Nonetheless as the functor $S$ uses the set $X$ to generate a symplectic vector space, each object in the image of $S$ has a canonical symplectomorphism with its dual $\Gamma_X : \mathbb{F}^X \oplus (\mathbb{F}^X)^* \to \mathbb{F}^X \oplus (\mathbb{F}^X)^*$, with name the Lagrangian subspace $\{ (\phi, i, \phi, -i) \mid \phi \in \mathbb{F}^X, i \in (\mathbb{F}^X)^* \}$. The image of the canonical self-duality Corel is thus given by the composite of this symplectomorphism with canonical duality in LagrRel; that is

$$S \left( \begin{array}{c} X \\ X \end{array} \right) = \bigcup_{\Gamma_X} S(X)$$

**Part III**

**The Black Box Functor**

We have now developed enough machinery to prove the main result of this paper: there is a monoidal dagger functor $\blacksquare : \text{Circ} \to \text{LagrRel}$ taking diagrams of passive linear circuits to their sets of behaviours. To recap, we have so far developed two categories: one capturing the physical form of circuits of passive linear components—in which the morphisms represent these circuits up to topological equivalence—and one capturing the behaviour of circuits of linear resistors—which contains morphisms representing circuits up to functional equivalence. We now define a functor that maps the physical form of a circuit to its behaviour. In particular, the composition rule for circuits by their function reflects the composition rule we use to define circuits by their form.

**8 Definition**

The role of the functor we construct here is to identify all circuits with the same behaviour on terminals, making the internal structure of the circuit inaccessible. Circuit components of this sort are frequently referred to as 'black boxes', and in acknowledgement to this we call this functor the **black box functor**, writing

$$\blacksquare : \text{Circ} \to \text{LagrRel}.$$  

On objects this functor maps finite sets to the symplectic vector space generated by this set. That is, let $X$ be an object of Circ. Then $X$ is a $(0, \infty)$-graph with no edges or, equivalently, a finite set. We define

$$\blacksquare(X) = \mathbb{F}^X \oplus (\mathbb{F}^X)^*.$$  

Defining the action of the black box functor on morphisms is a bit more involved. Let $\Gamma : X \to Y$ be a circuit. Recall that this means that $X$ and $Y$ are finite sets considered as $\mathbb{F}$-graphs with no edges, and $\Gamma$ is a $\mathbb{F}$-graph $(N, E, s, t, r)$ equipped with maps of $\mathbb{F}$-graphs $i : X \to \Gamma$ and $o : Y \to \Gamma$:
As before, we abuse notation by writing \( \Gamma \) for both the cospan and the apex of the cospan. To define the image of \( \Gamma \) under our functor \( \blacksquare \), by definition a Lagrangian relation \( \blacksquare(\Gamma) : \blacksquare(X) \to \blacksquare(Y) \), we must specify a Lagrangian subspace
\[
\blacksquare(\Gamma) \subseteq \mathbb{F}^X \oplus (\mathbb{F}^X)^* \oplus \mathbb{F}^Y \oplus (\mathbb{F}^Y)^*.
\]
Constructing this subspace is a matter of composing four processes we have discussed in the preceding sections.

Recall that to each \( \mathbb{F} \)-graph \( \Gamma \) we associate a Dirichlet form, the extended power functional
\[
P_\Gamma : \mathbb{F}^N \longrightarrow \mathbb{F};
\]
\[
\phi \longmapsto \sum_{e \in E} \frac{1}{r(e)} (\phi(t(e)) - \phi(s(e)))^2,
\]
and to this Dirichlet form we associate a Lagrangian subspace
\[
L_\Gamma = \{(\phi, d(P_\Gamma)\phi) \mid \phi \in \mathbb{F}^N \} \subseteq \mathbb{F}^N \oplus (\mathbb{F}^N)^*.
\]
From the legs of the cospan \( \Gamma \), the ideal wire functor \( S \) gives the Lagrangian relation
\[
S([i, o])^\dagger : \mathbb{F}^N \oplus (\mathbb{F}^N)^* \longrightarrow \mathbb{F}^X \oplus (\mathbb{F}^X)^* \oplus \mathbb{F}^Y \oplus (\mathbb{F}^Y)^*.
\]
Lastly, we have the symplectomorphism
\[
1_X : \mathbb{F}^X \oplus (\mathbb{F}^X)^* \longrightarrow \mathbb{F}^X \oplus (\mathbb{F}^X)^*;
\]
\[
(\phi, i) \longmapsto (\phi, -i).
\]
The black box functor maps a circuit \( \Gamma \) to the Lagrangian relation
\[
(1_X \oplus 1_Y) \circ S([i, o])^\dagger (L_\Gamma).
\]
This is well-typed by Proposition 6.8.

We picture this as

As isomorphisms of cospans of \((0, \infty)\)-graphs amount to no more than a relabelling of nodes and edges, this construction is independent of the cospan chosen as representative of the isomorphism class of cospans forming the circuit.

9 Proof of functoriality

To prove that the black box construction is indeed, as claimed, functorial, we factor it into three functors. These functors are each strong monoidal dagger functors, so the black box functor is too.
We first make use, two times over, of our results on decorated cospans in Section 4, showing the existence of categories of cospans decorated by Dirichlet forms and then Lagrangian subspaces, and the existence of functors from the category of circuits to each of these. This proceeds by defining functors Dirich and Lagr describing decorations and applying Lemma 4.2 to construct the corresponding cospan categories, and then by defining the relevant natural transformations and applying Lemma 4.5 to construct the desired functors.

The third functor takes cospans decorated by Lagrangian subspaces, and black boxes them to give Lagrangian relations between the feet of such cospans. The functoriality of this process relies on interpreting corelations as Lagrangian relations.

This gives a factorisation

\[
\begin{align*}
\circledast : \text{Circ} & \longrightarrow \text{DirichCospan} & \longrightarrow & \text{LagrCospan} & \longrightarrow & \text{LagrRel}
\end{align*}
\]

The following subsections deal with these functors in sequence, defining and proving the existence of each one using the techniques of Part II.

### 9.1 From circuits to Dirichlet cospans

#### The category of Dirichlet cospans

Let

\[
\text{Dirich} : (\text{FinSet}, +) \longrightarrow (\text{Set}, \times)
\]

map a finite set \( X \) to the set \( \text{Dirich}(X) \) of Dirichlet forms on \( X \), and map a function \( f : X \rightarrow Y \) between finite sets to the ‘pushforward’ function

\[
\text{Dirich}(f) : \text{Dirich}(X) \rightarrow \text{Dirich}(Y);
\]

\[
Q \mapsto (f_\ast Q : F^Y \rightarrow F; \phi \mapsto Q(\phi \circ f)).
\]

This defines a functor.

Moreover, equipping this functor with the family of maps

\[
\delta_{N,M} : \text{Dirich}(N) \times \text{Dirich}(M) \longrightarrow \text{Dirich}(N + M);
\]

\[
(Q_N, Q_M) \mapsto Q_N + Q_M,
\]

and unit

\[
\delta_1 : 1 \longrightarrow \text{Dirich}(\emptyset);
\]

\[
\bullet \ssr{\rightarrow} (\mathbb{R}^\emptyset \rightarrow \mathbb{F}; \emptyset \mapsto 0)
\]

defines a lax monoidal functor.

By Lemma 4.2, we thus have a category \( \text{DirichCospan} \) with morphisms cospans with apices equipped with Dirichlet forms. While a Dirichlet form captures the behaviour of a circuit, it does not allow for discussion of circuit start and end terminals. This cospan construction extends our framework to include these terminals.

In particular, note that we have overcome our inability to define a composition rule on Dirichlet forms. The zero Dirichlet form on \( \mathbb{R}^S \) acts as the identity morphism on \( S \). This corresponds to the fact that perfect conductors draw no power.

#### The functor \( \text{Circ} \rightarrow \text{DirichCospan} \)

We have now constructed two monoidal functors

\[
(\text{Circuit}, \rho), (\text{Dirich}, \delta) : (\text{FinSet}, +) \longrightarrow (\text{Set}, \times)
\]
that describe the circuit structures and Dirichlet forms we may put on the set respectively. We have also seen, motivating our discussion of Dirichlet forms, that from any circuit we can obtain a Dirichlet form describing the power usage of that circuit. This process respects the monoidal product: it specifies a monoidal natural transformation between Circuit and Dirich. By Lemma 4.5, this gives us a strict monoidal dagger functor $\text{Circ} \to \text{DirichCospan}$.

Define

$$\alpha : (\text{Circuit}, \rho) \Rightarrow (\text{Dirich}, \delta)$$

to be the collection of functions

$$\alpha_N : \text{Circuit}(N) \to \text{Dirich}(N):$$

$$(N, E, s, t, r) \mapsto \left( \phi \in k^N \mapsto \sum_{e \in E} (r(e))^{-1}(\phi(s(e)) - \phi(t(e))) \right)$$

We check naturality and monoidality.

Naturality requires that the square

$$\begin{array}{ccc}
\text{Circuit}(N) & \xrightarrow{\alpha_N} & \text{Dirich}(N) \\
\downarrow \text{Circuit}(f) & & \downarrow \text{Dirich}(f) \\
\text{Circuit}(M) & \xrightarrow{\alpha_M} & \text{Dirich}(M)
\end{array}$$

commutes. Let $(N, E, s, t, r)$ be a $(0, \infty)$-graph on $N$ and $f : N \to M$ be a function $N$ to $M$. Then both $\text{Dirich}(f) \circ \alpha_N$ and $\alpha_M \circ \text{Circuit}(f)$ map the graph $(N, E, s, t, r)$ to the Dirichlet form

$$k^M \to k;$$

$$\psi \mapsto \sum_{e \in E} r(e)^{-1}(\psi(f(s(e))) - \psi(f(t(e)))).$$ 

Thus both methods of constructing a power functional on a set of nodes $M$ from a circuit on $N$ and a function $N \to M$ produce the same power functional.

Furthermore, this method respects the monoidal structure on graphs and Dirichlet forms. That is, monoidality requires that the square

$$\begin{array}{ccc}
\text{Circuit}(N) \times \text{Circuit}(M) & \xrightarrow{\alpha_N \times \alpha_M} & \text{Dirich}(N) \times \text{Dirich}(M) \\
\downarrow \rho_{N,M} & & \downarrow \delta_{N,M} \\
\text{Circuit}(N + M) & \xrightarrow{\alpha_{N+M}} & \text{Dirich}(N + M)
\end{array}$$

and the triangle

$$\begin{array}{ccc}
\text{Circuit}(\emptyset) & \xrightarrow{\alpha_{\emptyset}} & \text{Dirich}(\emptyset) \\
\downarrow \rho_{\emptyset} & & \downarrow \delta_{\emptyset} \\
1 & & 1
\end{array}$$

commute. It is readily observed that both paths around the square lead to taking two graphs and summing their corresponding Dirichlet forms, and that the triangle commutes immediately as all objects in it are the one element set.

From Lemma 4.5, we thus obtain a strict monoidal dagger functor

$$Q : \text{Circ} = \text{CircuitCospan} \to \text{DirichCospan}.$$
Roughly, this says that the process of composition for circuit diagrams is the same as that of composition for Dirichlet cospans. Note that although this functor preserves much of the information in circuit diagrams, it is not a faithful functor. The effect of this functor is to remove parallel edges and paths within a graph.

9.2 From Dirichlet cospans to Lagrangian cospans

The category of Lagrangian cospans

We now define a functor

\[ \text{Lagr} : (\text{FinSet}, +) \to (\text{Set}, \times). \]

On objects, let this map a finite set \( X \) to the set \( \text{Lagr}(X) \) of Lagrangian subspaces of the symplectic vector space \( \mathbb{F}^X \oplus (\mathbb{F}^X)^* \). For morphisms, recall that a function \( f : X \to Y \) between finite sets may be considered as a correlation, and the functor \( S \) of Section 7 thus maps this correlation to some Lagrangian relation \( S(f) : \mathbb{F}^X \oplus (\mathbb{F}^X)^* \to \mathbb{F}^Y \oplus (\mathbb{F}^Y)^* \). As Lagrangian relations map Lagrangian subspaces to Lagrangian subspaces (Proposition 6.8), this gives a map:

\[ \text{Lagr}(f) : \text{Lagr}(X) \to \text{Lagr}(Y); \]

\[ L \mapsto S(f)(L), \]

The functoriality of this construction follows from the functoriality of \( S \).

Again moreover, equipping this functor with the family of maps

\[ \lambda_{N,M} : \text{Lagr}(N) \times \text{Lagr}(M) \to \text{Lagr}(N + M); \]

\[ (L_N, L_M) \mapsto L_N \oplus L_M, \]

and unit

\[ \lambda_1 : 1 \to \text{Lagr}(\emptyset); \]

\[ \bullet \mapsto \{0\} \]

defines a lax monoidal functor. We thus have a category \( \text{LagrCospan} \).

The functor \( \text{DirichCospan} \to \text{LagrCospan} \)

We now wish to construct a strict monoidal dagger functor \( \text{DirichCospan} \to \text{LagrCospan} \) via Lemma 4.5 and a monoidal natural transformation between the monoidal functors

\[ (\text{Dirich}, \delta), (\text{Lagr}, \lambda) : (\text{FinSet}, +) \to (\text{Set}, \times). \]

To this end, define

\[ \beta : (\text{Dirich}, \delta) \Rightarrow (\text{Lagr}, \lambda) \]

to be the collection of functions

\[ \beta_N : \text{Dirich}(N) \to \text{Lagr}(N); \]

\[ Q \mapsto \{ (\phi, dQ_{\phi}) \mid \phi \in \mathbb{F}^N \} \subseteq \mathbb{F}^N \oplus (\mathbb{F}^N)^*. \]

Naturality requires that the square

\[
\begin{array}{ccc}
\text{Dirich}(N) & \xrightarrow{\beta_N} & \text{Lagr}(N) \\
\text{Dirich}(f) \downarrow & & \downarrow \text{Lagr}(f) \\
\text{Dirich}(M) & \xrightarrow{\beta_M} & \text{Lagr}(M)
\end{array}
\]

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commutes for every function $f : N \to M$. This is primarily a consequence of the fact that the exterior derivative commutes with pullbacks. Following the potentials functor defined in Subsection 7.3, write $\Phi f$ for the linear map $F^M \to F^N; \psi \mapsto \psi \circ f$. Then $\text{Dirich}(f)$ maps a Dirichlet form $Q$ on $N$ to the form $Q \circ \Phi f$, and $\beta_M$ in turn maps this to the Lagrangian subspace comprising vectors of the form $(\psi, d(Q \circ \Phi f) \psi)$, where $\psi \in F^M$. On the other hand, $\beta_N$ maps a Dirichlet form $Q$ on $N$ to the Lagrangian subspace comprising vectors of the form $(\psi, (\Phi f)^*dQ \phi)$, where $\psi$ obeys $\psi = \Phi f(\phi)$. But

$$(\Phi f)^*dQ \phi = d(Q \circ \Phi f) \psi,$$

so these two processes commute.

Monoidality requires that the diagrams

$$
\begin{array}{ccc}
\text{Dirich}(N) \times \text{Dirich}(M) & \xrightarrow{\beta_N \times \beta_M} & \text{Lagr}(N) \times \text{Lagr}(M) \\
\delta_{N,M} \downarrow & & \downarrow \lambda_{N,M}
\end{array}
\quad \text{and}
\begin{array}{ccc}
\text{Dirich}(N + M) & \xrightarrow{\beta_{N+M}} & \text{Lagr}(N + M) \\
\delta_{\emptyset} \downarrow & & \downarrow \lambda_{\emptyset}
\end{array}
$$

commute. These do: the Lagrangian subspace corresponding to the sum of Dirichlet forms is equal to the sum of the Lagrangian subspaces that correspond to the summand Dirichlet forms, while there is only a unique map $1 \to \text{Lagr}(\emptyset)$.

From Lemma 4.5, we thus obtain a strict monoidal dagger functor.

### 9.3 From cospans to relations

At this point we have checked that the process of reinterpreting a circuit as a Lagrangian subspace of behaviours is functorial. Our task is now to integrate this information as more than just a ‘decoration’ on our morphisms. This process constitutes a strong monoidal dagger functor

$$\text{LagrCospan} \to \text{LagrRel}.$$ 

This factor of the black box functor is the factor that gives it its name; through this functor we finally seal off the inner structure of our circuits, giving us access only to the behaviour at the terminals. Its purpose is to take a Lagrangian cospan, which captures information about the behaviours of a circuit measured at each terminal and internal node, and restrict it down to a relation detailing the behaviours simply on the terminals.

On objects this functor takes, finally, the finite sets $X$ of $\text{LagrCospan}$ to the symplectic vector spaces $\mathbb{F}^X \oplus (\mathbb{F}^X)^*$ of possible potential-current readings at the points of these sets. On morphisms they take a Lagrangian cospan

$$(X \xrightarrow{i} N \xleftarrow{\alpha} Y; L \subseteq \mathbb{F}^N \oplus (\mathbb{F}^N)^*)$$

to the Lagrangian relation

$$(\Gamma_X \oplus 1_Y) \circ S([i, \alpha])^\dagger \circ L \subseteq \mathbb{F}^X \oplus (\mathbb{F}^X)^* \oplus \mathbb{F}^Y \oplus (\mathbb{F}^Y)^*.$$ 

This Lagrangian relation is of the same form as the image of circuits under the black box functor.

Inspection shows that this map preserves identities and the dagger functors. Furthermore, as there are natural symplectomorphisms

$$\mathbb{F}^X \oplus (\mathbb{F}^X)^* \oplus \mathbb{F}^Y \oplus (\mathbb{F}^Y)^* \xrightarrow{\sim} \mathbb{F}^{X+Y} \oplus (\mathbb{F}^{X+Y})^*$$

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and a symplectomorphism
\[ \{0\} \xrightarrow{\sim} \mathbb{F}^\mathbb{F} \oplus (\mathbb{F}^\mathbb{F})^\ast = \{0\} \]
the map preserves monoidal composition. It thus remains to check that composition is preserved.
As usual, this takes a moment. In terms of names, this comes down to checking the equality of Lagrangian subspaces

\[
\begin{align*}
& L[X] \quad K[Y] \\
\downarrow \quad \downarrow \\
\mathcal{N} & \quad \mathcal{M}
\end{align*}
\]

where the left hand side is the composite in LagrRel of the images of the Lagrangian cospans

\[
(X \xrightarrow{i_X} N \xleftarrow{o_Y} Y; L) \quad \text{and} \quad (Y \xrightarrow{i_Y} M \xleftarrow{o_Z} Z; K),
\]

and the right hand side is the image of their composite in LagrCospan.

Recalling Remark 7.5, regarding the functor \(S\) and duals for objects, this implies that it is enough to check the equality of corelations

\[
\begin{align*}
& N[X] \quad M[Z] \\
\downarrow \quad \downarrow \\
\mathcal{N} & \quad \mathcal{M}
\end{align*}
\]

Writing this instead as a commutative diagram, we wish to prove the equality of the two corelations \(N + M \rightarrow X + Y\):

\[
\begin{align*}
& N + M \\
\downarrow \quad \downarrow \\
\mathcal{N} & \quad \mathcal{M}
\end{align*}
\]

As we have the factorisation
\[
[j_N \circ i_X, j_M \circ o_Z] = [j_N \circ i_X, j_N \circ o_Y, j_M \circ o_Z] \circ \text{incl}_{X + Z} : X + Z \rightarrow X + Y + Z \rightarrow N + Y M,
\]
it is enough to check the following diagram commutes from \(N + M\) to \(X + Y + Z\) in the category of corelations:

\[
\begin{align*}
& N + M \\
\downarrow \quad \downarrow \\
\mathcal{N} & \quad \mathcal{M}
\end{align*}
\]

By Lemma 7.4, this is equivalent to checking that it is a pushout square in FinSet. This is so; the square commutes in FinSet as it is the sum along the lower right edges of the three commutative
and given any other object $T$ and maps $f, g$ such that

\[
\begin{align*}
\begin{array}{c}
\text{commutes, there is a unique map } N + Y M \to T \text{ defined by sending } a \text{ in } N + Y M \text{ to } f(\hat{a}), \text{ where } \\
\hat{a} \text{ is a preimage of } a \text{ in } N + M \text{ under the coproduct of pushout maps } [j_N, j_M]. \text{ This is well-defined as the preimage of } a \text{ is either unique or equal to } \\
\{o_Y(y), i_Y(y)\} \text{ for some element } y \in Y, \text{ and the } \\
\text{commutativity of the above square containing } T \text{ implies that } f(o_Y(y)) = f(i_Y(y)). \text{ This proves the } \\
\text{functoriality of the map } \text{LagrCospan} \to \text{LagrRel} \text{ defined above.}
\end{array}
\end{align*}
\]

The three functors of this section compose to the black box functor, thus proving its functoriality.

References


