Classical Mechanics Homework
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1 The 2-Body Problem

The goal of this problem is to understand a pair of particles interacting via a central force such as gravity. We'll reduce it to problem you've already studied — the case of a *single* particle in a central force.

Suppose we have a system of two particles interacting by a central force. Their positions are functions of time, say $q_1, q_2 \colon \mathbb{R} \to \mathbb{R}^3$, satisfying Newton's law:

$$m_1 \ddot{q}_1 = f(|q_1 - q_2|) \frac{q_1 - q_2}{|q_1 - q_2|}$$

$$m_2 \ddot{q}_2 = f(|q_2 - q_1|) \frac{q_2 - q_1}{|q_2 - q_1|}.$$

Here m_1, m_2 are their masses, and the force is described by some smooth function $f \colon (0, \infty) \to \mathbb{R}$. Let's write the force in terms of a potential as follows:

$$f(r) = -\frac{dV}{dr}.$$

Using conservation of momentum and symmetry under translations and Galilei boosts we can work in coordinates where

$$m_1 q_1(t) + m_2 q_2(t) = 0 \tag{1}$$

for all times t. This coordinate system is called the **center-of-mass frame**.

We could use equation (1) to solve for q_2 in terms of q_1, or vice versa, but we can also use it to express both q_1 and q_2 in terms of the **relative position**

$$q(t) = q_1(t) - q_2(t).$$

This is more symmetrical, so this is what we will do. Henceforth we only need to talk about q. Thus we have reduced the problem to a 1-body problem!

Now here's where you come in:

1. Show that $q(t)$ satisfies the equation

$$m\ddot{q} = f(|q|) \frac{q}{|q|}$$

where m is the so-called **reduced mass**

$$m = \frac{m_1 m_2}{m_1 + m_2}.$$

Note that this looks exactly like Newton's second law for a single particle!

2. Recall that the total energy E of the 2-particle system is the sum of the kinetic energies of the particles plus the potential energy. Express E in terms of q and the reduced mass. Show that

$$E = \frac{1}{2} m |\dot{q}|^2 + V(|q|)$$

Note that this looks exactly like the energy of a single particle!

3. Let J be the total angular momentum of the 2-particle system. Show that

$$J = mq \times \dot{q}$$

Note that this looks exactly like the angular momentum of a single particle!

At this point we're back to a problem you've already solved: a *single* particle in a central force. The only difference is that now q stands for the *relative* position and m stands for the *reduced* mass!

So, we instantly conclude that two bodies orbiting each other due to the force of gravity will *both* have an orbit that's either an ellipse, or a parabola, or a hyperbola... when viewed in the center-of-mass frame.

2 Poisson brackets

Let $\mathbb{R}^{2n}$ be the **phase space** of a particle in $\mathbb{R}^n$, with coordinates q_i, p_i $(1 \le i \le n)$. Let $C^\infty(\mathbb{R}^{2n})$ be the set of smooth real-valued functions on $\mathbb{R}^{2n}$, which becomes an commutative algebra using pointwise addition and multiplication of functions.

We define the **Poisson bracket** of functions $F, G \in C^\infty(\mathbb{R}^{2n})$ by:

$$\{F, G\} = \sum_{i=1}^{n} \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i} - \frac{\partial G}{\partial p_i}\frac{\partial F}{\partial q_i}.$$

4. Show that Poisson brackets make the vector space $C^\infty(R^{2n})$ into a **Lie algebra**. In other words, check the **antisymmetry** of the bracket:

$$\{F, G\} = -\{G, F\}$$

the **bilinearity** of the bracket:

$$\{F, \alpha G + \beta H\} = \alpha\{F, G\} + \beta\{F, H\}$$

$$\{\alpha F + \beta G, H\} = \alpha\{F, H\} + \beta\{G, H\}$$

and **Jacobi identity**:

$$\{F, \{G, H\}\} = \{\{F, G\}, H\} + \{G, \{F, H\}\}$$

for all $F, G, H \in C^\infty(\mathbb{R}^{2n})$ and $\alpha, \beta \in \mathbb{R}$.

(*Note the Jacobi identity resembles the product rule $d(GH) = (dG)H + GdH$, with bracketing by F playing the role of d. This is no accident!*)

5. Show that Poisson brackets and ordinary multiplication of functions make the vector space $C^\infty(\mathbb{R}^{2n})$ into a **Poisson algebra**. This is a Lie algebra that is also a commutative algebra, with the bracket $\{F, G\}$ and the product FG related by the **Leibniz identity**:

$$\{F, GH\} = \{F, G\}H + G\{F, H\}.$$

(*Again this identity resembles the product rule!*)