1 The 2-Body Problem

The goal of this problem is to understand a pair of particles interacting via a central force such as gravity. We'll reduce it to problem you've already studied — the case of a single particle in a central force.

Suppose we have a system of two particles interacting by a central force. Their positions are functions of time, say \( q_1, q_2 : \mathbb{R} \to \mathbb{R}^3 \), satisfying Newton's law:

\[
\begin{align*}
    m_1 \ddot{q}_1 &= f(|q_1 - q_2|) \frac{q_1 - q_2}{|q_1 - q_2|} \\
    m_2 \ddot{q}_2 &= f(|q_2 - q_1|) \frac{q_2 - q_1}{|q_2 - q_1|}.
\end{align*}
\]

Here \( m_1, m_2 \) are their masses, and the force is described by some smooth function \( f : (0, \infty) \to \mathbb{R} \).

Let's write the force in terms of a potential as follows:

\[
f(r) = -\frac{dV}{dr}
\]

Using conservation of momentum and symmetry under translations and Galilei boosts we can work in coordinates where

\[
m_1 q_1(t) + m_2 q_2(t) = 0
\]

for all times \( t \). This coordinate system is called the center-of-mass frame.

We could use equation (1) to solve for \( q_2 \) in terms of \( q_1 \), or vice versa, but we can also use it to express both \( q_1 \) and \( q_2 \) in terms of the relative position

\[
q(t) = q_1(t) - q_2(t).
\]

This is more symmetrical, so this is what we will do. Henceforth we only need to talk about \( q \). Thus we have reduced the problem to a 1-body problem!

Now here's where you come in:

1. Show that \( q(t) \) satisfies the equation

\[
m \dot{q} = f(|q|) \frac{q}{|q|}
\]

where \( m \) is the so-called reduced mass

\[
m = \frac{m_1 m_2}{m_1 + m_2}.
\]

Note that this looks exactly like Newton's second law for a single particle!

2. Recall that the total energy \( E \) of the 2-particle system is the sum of the kinetic energies of the particles plus the potential energy. Express \( E \) in terms of \( q \) and the reduced mass. Show that

\[
E = \frac{1}{2} m |\dot{q}|^2 + V(|q|)
\]
Note that this looks exactly like the energy of a single particle!

3. Let $J$ be the total angular momentum of the 2-particle system. Show that

$$J = mq \times \dot{q}$$

Note that this looks exactly like the angular momentum of a single particle!

At this point we’re back to a problem you’ve already solved: a single particle in a central force. The only difference is that now $q$ stands for the relative position and $m$ stands for the reduced mass!

So, we instantly conclude that two bodies orbiting each other due to the force of gravity will both have an orbit that’s either an ellipse, or a parabola, or a hyperbola... when viewed in the center-of-mass frame.

2 Poisson brackets

Let $\mathbb{R}^{2n}$ be the phase space of a particle in $\mathbb{R}^n$, with coordinates $q_i, p_i$ ($1 \leq i \leq n$). Let $C^\infty(\mathbb{R}^{2n})$ be the set of smooth real-valued functions on $\mathbb{R}^{2n}$, which becomes an commutative algebra using pointwise addition and multiplication of functions.

We define the Poisson bracket of functions $F, G \in C^\infty(\mathbb{R}^{2n})$ by:

$$\{F, G\} = \sum_{i=1}^n \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} - \frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i}.$$ 

4. Show that Poisson brackets make the vector space $C^\infty(\mathbb{R}^{2n})$ into a Lie algebra. In other words, check the antisymmetry of the bracket:

$$\{F, G\} = -\{G, F\}$$

the bilinearity of the bracket:

$$\{F, \alpha G + \beta H\} = \alpha \{F, G\} + \beta \{F, H\}$$

$$\{\alpha F + \beta G, H\} = \alpha \{F, H\} + \beta \{G, H\}$$

and Jacobi identity:

$$\{F, \{G, H\}\} = \{\{F, G\}, H\} + \{G, \{F, H\}\}$$

for all $F, G, H \in C^\infty(\mathbb{R}^{2n})$ and $\alpha, \beta \in \mathbb{R}$.

(Note the Jacobi identity resembles the product rule $d(GH) = (dG)H + GdH$, with bracketing by $F$ playing the role of $d$. This is no accident!)

5. Show that Poisson brackets and ordinary multiplication of functions make the vector space $C^\infty(\mathbb{R}^{2n})$ into a Poisson algebra. This is a Lie algebra that is also a commutative algebra, with the bracket $\{F, G\}$ and the product $FG$ related by the Leibniz identity:

$$\{F, GH\} = \{F, G\} H + G\{F, H\}.$$ 

(Again this identity resembles the product rule!)