Classical Mechanics Homework
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The 2-Body Problem

Suppose we have a system of two particles interacting by a central force. Their positions are functions of time, say $q_1, q_2 \colon \mathbb{R} \to \mathbb{R}^3$, satisfying Newton's law:

$$m_1 \ddot{q}_1 = f(|q_1 - q_2|) \frac{q_1 - q_2}{|q_1 - q_2|}$$

$$m_2 \ddot{q}_2 = f(|q_2 - q_1|) \frac{q_2 - q_1}{|q_2 - q_1|}.$$

Here m_1, m_2 are their masses, and the force is described by some smooth function $f \colon (0, \infty) \to \mathbb{R}$. Let's write the force in terms of a potential as follows:

$$f(r) = -\frac{dV}{dr}.$$

Using conservation of momentum and symmetry under translations and Galilei boosts we can work in coordinates where

$$m_1 q_1(t) + m_2 q_2(t) = 0 \qquad (1)$$

for all times t. This coordinate system is called the **center-of-mass frame**.

We could use equation (1) to solve for q_2 in terms of q_1, or vice versa, but we can also use it to express both q_1 and q_2 in terms of the **relative position**

$$q(t) = q_1(t) - q_2(t).$$

This is more symmetrical, so this is what we will do. Henceforth we only need to talk about q. Thus we have reduced the problem to a 1-body problem!

1. Show that $q(t)$ satisfies the equation

$$m\ddot{q} = f(|q|) \frac{q}{|q|}$$

where m is the so-called **reduced mass**

$$m = \frac{m_1 m_2}{m_1 + m_2}.$$

Using the expressions for $m_i \ddot{q}_i$ in terms of f given in the introduction, we have:

$$\begin{aligned} m\ddot{q} &= \frac{m_1 m_2}{m_1 + m_2} (\ddot{q}_1 - \ddot{q}_2) \\ &= \frac{1}{m_1 + m_2} \left(m_2 f(|q|) \frac{q}{|q|} + m_1 f(|q|) \frac{q}{|q|} \right) \\ &= f(|q|) \frac{q}{|q|}. \end{aligned}$$

2. Show that

$$E = \frac{1}{2}m|\dot{q}|^2 + V(|q|).$$

Recall that the the kinetic energy of the i-th particle is given by

$$T_i = \frac{1}{2}m\dot{q}_i^2$$

and that the total kinetic energy for the system is $T = T_1 + T_2$. Bearing this in mind, we calculate:

$$\begin{aligned}
\frac{1}{2}m\dot{q}^2 &= \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}(\dot{q}_1 - \dot{q}_2)^2 \\
&= \frac{1}{m_1 + m_2}(m_2 T_1 + m_1 T_2 - m_1 m_2 \dot{q}_1 \dot{q}_2). \qquad (2)
\end{aligned}$$

Note that (1) implies that

$$\dot{q}_i = -\frac{m_j}{m_i}\dot{q}_j \qquad (3)$$

for q_i equal to q_1 or q_2. Hence

$$\frac{1}{2}m_1 m_2 \dot{q}_1 \dot{q}_2 = -\frac{1}{2}m_i^2 \dot{q}_i^2 = -m_i T_i$$

for $i = 1, 2$. Thus (2) becomes

$$\frac{1}{m_1 + m_2}(m_2 T_1 + m_1 T_2 - m_1 m_2 \dot{q}_1 \dot{q}_2) = \frac{1}{m_1 + m_2}\left(m_2 T_1 + m_1 T_2 + (m_1 T_1 + m_2 T_2)\right) = T_1 + T_2.$$

Whence

$$\frac{1}{2}m\dot{q}^2 = T_1 + T_2 = T,$$

and the energy of the system is given by

$$E = T + V(|q_1 - q_2|) = \frac{1}{2}m\dot{q}^2 + V(|q|).$$

3. Establish

$$J = mq \times \dot{q}$$

where J is the total angular momentum.

We work from the right hand side:

$$
\begin{aligned}
mq \times \dot{q} &= \frac{m_1 m_2}{m_1+m_2}(q_1 - q_2) \times (\dot{q}_1 - \dot{q}_2) \\
&= \frac{m_1 m_2}{m_1+m_2}(q_1 \times \dot{q}_1 - q_1 \times \dot{q}_2 - q_2 \times \dot{q}_1 + q_2 \times \dot{q}_2) \\
&= \frac{m_1 m_2}{m_1+m_2}\left(q_1 \times \dot{q}_1 + \frac{m_1}{m_2} q_1 \times \dot{q}_1 + \frac{m_2}{m_1} q_2 \times \dot{q}_2 + q_2 \times \dot{q}_2\right) \\
&= \frac{m_1 m_2}{m_1+m_2}\left(\frac{m_1+m_2}{m_2} q_1 \times \dot{q}_1 + \frac{m_1+m_2}{m_1} q_2 \times \dot{q}_2\right) \\
&= m_1 q_1 \times \dot{q}_1 + m_2 q_2 \times \dot{q}_2 \\
&= J_1 + J_2 \\
&= J;
\end{aligned}
$$

where the third equality follows from (3).

Poisson brackets

Let $\mathbb{R}^{2n}$ be the **phase space** of a particle in $\mathbb{R}^n$, with coordinates q_i, p_i $(1 \le i \le n)$. Let $C^\infty(\mathbb{R}^{2n})$ be the set of smooth real-valued functions on $\mathbb{R}^{2n}$, which becomes an commutative algebra using pointwise addition and multiplication of functions.

We define the **Poisson bracket** of functions $F, G \in C^\infty(\mathbb{R}^{2n})$ by:

$$\{F, G\} = \sum_{i=1}^{n} \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i} - \frac{\partial G}{\partial p_i}\frac{\partial F}{\partial q_i}.$$

4. Show that Poisson brackets make the vector space $C^\infty(R^{2n})$ into a **Lie algebra**. In other words, check the **antisymmetry** of the bracket:

$$\{F, G\} = -\{G, F\}$$

the **bilinearity** of the bracket:

$$\{F, \alpha G + \beta H\} = \alpha\{F, G\} + \beta\{F, H\}$$
$$\{\alpha F + \beta G, H\} = \alpha\{F, H\} + \beta\{G, H\}$$

and **Jacobi identity**:

$$\{F, \{G, H\}\} = \{\{F, G\}, H\} + \{G, \{F, H\}\}$$

for all $F, G, H \in C^\infty(\mathbb{R}^{2n})$ and $\alpha, \beta \in \mathbb{R}$.

To make life a whole lot easier on ourselves we will impose the following conventions:

1. Lower indices indicate differentiation with respect to p_i, for instance:

$$F_i = \frac{\partial F}{\partial p_i}.$$

2. Upper indices indicate differentiation with respect to q_i, that is:

$$G^i = \frac{\partial G}{\partial q_i}.$$

3. All indexed quantities are assumed to represent sums. For instance:

$$F^i_j G_i = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial F}{\partial p_j \partial q_i}\frac{\partial G}{\partial i}.$$

Now then, let's get down to business. First, we will show that $\{\cdot,\cdot\}$ anti-commutes. This is simple, since

$$\{F,G\} = F_i G^i - G_i F^i = -(G_i F^i - F_i G^i) = -\{G,F\}.$$

Bilinearity is likewise trivial:

$$\{F, aG+bH\} = F_i(aG+bH)^i - (aG+bH)_i F^i = a(F_i G^i - G_i F^i) + b(F_i H^i - H_i F^i) = a\{F,G\} + b\{F,H\}$$

(*and similarly for the right hand slot of* $\{\cdot,\cdot\}$).

Now for the rough part of establishing the Jacobi identity. We will see that our conventions will reduce the problem to one of accounting—surely there is a more elegant way to establish the identity, but sometimes it is nice to get our hands dirty with actual computation. We will write down only the significant portions of the computation, leaving the reader to fill in the gaps. Here we go:

$$\begin{aligned}\{\{F,G\},H\} + \{G,\{F,H\}\} &= \{F,G\}_i H^i - H_i\{F,G\}^i + G_i\{F,H\}^i - \{F,H\}_i G^i \\ &= (F_j G^j - G_j F^j)_i H^i - H_i(F_j G^j - G_j F^j)^i + G_i(F_j H^j - H_j F^j)^i \\ &\qquad - (F_j H^j - H_j F^j)_i G^i.\end{aligned}$$

At this stage, we complete the differentiation and collect first and second partials of F (using the fact that F is smooth—i.e.- the mixed second partials of F are equal) to obtain:

$$\begin{aligned}&F_{ij}(G^j H^i - G^i H^j) + F^i_j[(G_i H^j - G_j H^i) + (G^i H_j - G^j H_i)] + F^{ij}(H_i G_j - H_j G_i) \\ &+F_j(G^j_i H^i - H_i G^{ij} + G_i H^{ij} - G^i H^j_i) + F^j(G^i_j H_i - G_i H^i_j + G^i H_{ij} - G_{ij} H^i).\end{aligned}$$

Now, the first three summands of the above vanish. To see this, note that if $i = j$ then each term is 0, and if $i \neq j$ exchanging i with j makes the terms negative: hence, in the sum, each of these three terms come in opposite pairs. Taking this cancellation into account, and noting that the last two terms of the sum are actually

$$F_j(G_i H^i - G^i H_i)^j - (G_i H^i - G^i H_i)_j F^j = F_j\{G,H\}^j - \{G,H\}_j F^j,$$

we see that

$$\{\{F,G\},H\} + \{G,\{F,H\}\} = \{F,\{G,H\}\}.$$

5. Show that Poisson brackets and ordinary multiplication of functions make the vector space $C^\infty(\mathbb{R}^{2n})$ into a **Poisson algebra**. This is a Lie algebra that is also a commutative algebra, with the bracket $\{F,G\}$ and the product FG related by the **Leibniz identity**:

$$\{F,GH\} = \{F,G\}H + G\{F,H\}.$$

Again, the conventions we established above will set us free:

$$\begin{aligned}\{F,GH\} &= F_i(GH)^i - (GH)_i F^i = F_i(G^i H + G H^i) - (G_i H + G H_i)F^i \\ &= (F_i G^i - G_i F^i)H + G(F_i H^i - H_i F^i) = \{F,G\}H + G\{F,H\}.\end{aligned}$$