Classical Mechanics Homework
January 30, 2008
John Baez
Homework Solutions by Michael Maroun

1 The 2-Body Problem

The goal of this problem is to understand a pair of particles interacting via a central force such as gravity. We'll reduce it to problem you've already studied — the case of a *single* particle in a central force.

Suppose we have a system of two particles interacting by a central force. Their positions are functions of time, say $q_1, q_2 \colon \mathbb{R} \to \mathbb{R}^3$, satisfying Newton's law:

$$m_1 \ddot{q}_1 = f(|q_1 - q_2|) \frac{q_1 - q_2}{|q_1 - q_2|}$$

$$m_2 \ddot{q}_2 = f(|q_2 - q_1|) \frac{q_2 - q_1}{|q_2 - q_1|}.$$

Here m_1, m_2 are their masses, and the force is described by some smooth function $f \colon (0, \infty) \to \mathbb{R}$. Let's write the force in terms of a potential as follows:

$$f(r) = -\frac{dV}{dr}.$$

Using conservation of momentum and symmetry under translations and Galilei boosts we can work in coordinates where

$$m_1 q_1(t) + m_2 q_2(t) = 0 \tag{1}$$

for all times t. This coordinate system is called the **center-of-mass frame**.

We could use equation (1) to solve for q_2 in terms of q_1, or vice versa, but we can also use it to express both q_1 and q_2 in terms of the **relative position**

$$q(t) = q_1(t) - q_2(t). \tag{2}$$

This is more symmetrical, so this is what we will do. Henceforth we only need to talk about q. Thus we have reduced the problem to a 1-body problem!

Now here's where you come in:

1. Show that $q(t)$ satisfies the equation

$$m\ddot{q} = f(|q|) \frac{q}{|q|}$$

where m is the so-called **reduced mass**

$$m = \frac{m_1 m_2}{m_1 + m_2}.$$

Note that this looks exactly like Newton's second law for a single particle!

Solution to 1:

From (2) we can immediately see that $\ddot{q} = \ddot{q}_1 - \ddot{q}_2$. We can re-write $\ddot{q}_1$ and $\ddot{q}_2$ using their respective equations of motion. This leaves us with:

$$\ddot{q}_1 = f(|q_1 - q_2|)\frac{q_1 - q_2}{m_1\,|q_1 - q_2|} \quad \text{and} \quad \ddot{q}_2 = f(|q_2 - q_1|)\frac{q_2 - q_1}{m_1\,|q_2 - q_1|}.$$

Subtracting these two equations we get an expression for $\ddot{q}$ as follows:

$$\ddot{q} = f(|q_1 - q_2|)\frac{q_1 - q_2}{m_1\,|q_1 - q_2|} - f(|q_2 - q_1|)\frac{q_2 - q_1}{m_2\,|q_2 - q_1|}.$$

Now since $|q_1 - q_2| = |q_2 - q_1|$ the above expression can be written as:

$$\ddot{q} = f(|q|)\frac{q_1 - q_2}{m_1\,|q|} - f(|q|)\frac{q_2 - q_1}{m_2\,|q|} = \frac{f(|q|)}{|q|}\left(\frac{q_1 - q_2}{m_1} - \frac{q_2 - q_1}{m_2}\right).$$

Combining q_i terms of equal i we get:

$$\ddot{q} = \frac{f(|q|)}{|q|}\left[\left(\frac{m_1 + m_2}{m_1\,m_2}\right)q_1 - \left(\frac{m_1 + m_2}{m_1\,m_2}\right)q_2\right].$$

The coefficients in front of q_1 and q_2 are just the inverse of the reduced mass.

$$\frac{1}{m} = \frac{m_1 + m_2}{m_1\,m_2}$$

Thus we have shown that:

$$\ddot{q} = \frac{1}{m}\frac{f(|q|)}{|q|}(q_1 - q_2)$$
$$\Rightarrow m\ddot{q} = f(|q|)\frac{q}{|q|}.$$

2. Recall that the total energy E of the 2-particle system is the sum of the kinetic energies of the particles plus the potential energy. Express E in terms of q and the reduced mass. Show that

$$E = \frac{1}{2}m\,|\dot{q}|^2 + V(|q|)$$

Note that this looks exactly like the energy of a single particle!

Solution to 2:

The total energy E for the two particle system is given by:

$$E = \frac{1}{2}m_1\dot{q}_1{}^2 + \frac{1}{2}m_2\dot{q}_2{}^2 + V(|q_1 - q_2|) \tag{3}$$

From (1) and (2) we express $\dot{q}_1$ and $\dot{q}_2$ in terms of $\dot{q}$, the mass m_1 and the mass m_2 by noting the following:

$$m_1\dot{q}_1 + m_2\dot{q}_2 = 0 \quad \Rightarrow \quad \dot{q}_1 = -\frac{m_2}{m_1}\dot{q}_2 \quad \text{and} \quad \dot{q}_2 = -\frac{m_1}{m_2}\dot{q}_1$$

$$\dot{q}_1 = -\frac{m_2}{m_1}\dot{q}_2 \quad \text{and} \quad \dot{q}_1 - \dot{q}_2 = \dot{q} \quad \Rightarrow \quad \dot{q}_2 + \frac{m_2}{m_1}\dot{q}_2 = -\dot{q} \quad \Rightarrow \quad \dot{q}_2 = \frac{-m_1}{m_1 + m_2}\dot{q} \tag{4a}$$

$$\dot{q}_2 = -\frac{m_1}{m_2}\dot{q}_1 \quad \text{and} \quad \dot{q}_1 - \dot{q}_2 = \dot{q} \quad \Rightarrow \quad \dot{q}_1 + \frac{m_1}{m_2}\dot{q}_1 = \dot{q} \quad \Rightarrow \quad \dot{q}_1 = \frac{m_2}{m_1 + m_2}\dot{q} \tag{4b}$$

Substituting the right most equalities of (4a) and (4b) for their respective values in (3) and noting that $|q_1 - q_2| = |q|$ we get:

$$E = \frac{1}{2}\frac{m_1\, m_2^2}{(m_1+m_2)^2}\,\dot{q}^{\,2} + \frac{1}{2}\frac{m_2 m_1^2}{(m_1+m_2)^2}\,\dot{q}^{\,2} + V(|q|). \tag{5}$$

Finally, summing the two kinetic terms explicitly gives the desired result.

$$E = \frac{1}{2}\, m\, |\dot{q}|^2 + V(|q|)$$

Where as before m is the reduced mass of the system.

3. Let J be the total angular momentum of the 2-particle system. Show that

$$J = mq \times \dot{q}$$

Note that this looks exactly like the angular momentum of a single particle!

Solution to 3:

Solving for the positions q_1 and q_2, gives identical equations to (4a) and (4b) only without time derivatives. We summarize the results below:

$$q_1 = \frac{m_2}{m_1+m_2}\, q \qquad \dot{q}_1 = \frac{m_2}{m_1+m_2}\, \dot{q} \tag{6a}$$

$$q_2 = \frac{-m_1}{m_1+m_2}\, q \qquad \dot{q}_2 = \frac{-m_1}{m_1+m_2}\, \dot{q} \tag{6b}$$

The total angular momentum, J, is the sum of the angular momentum of each particle. We give the full expression below.

$$J = J_1 + J_2 = m_1\, q_1 \times \dot{q}_1 + m_2\, q_2 \times \dot{q}_2 \tag{7}$$

Again, inserting (6a) and (6b) into (7) in favor of q we get:

$$\begin{aligned}
J &= m_1\, q_1 \times \dot{q}_1 + m_2\, q_2 \times \dot{q}_2 \\
&= m_1\, \frac{m_2}{m_1+m_2}\, q \times \frac{m_2}{m_1+m_2}\, \dot{q} + m_2\, \frac{-m_1}{m_1+m_2} \times \frac{-m_1}{m_1+m_2}\, \dot{q} \\
&= \frac{m_1\, m_2^2}{(m_1+m_2)^2}\, q \times \dot{q} + \frac{m_2\, m_1^2}{(m_1+m_2)^2} \times \dot{q} \\
&= \frac{m_1\, m_2}{(m_1+m_2)}\, \frac{m_1+m_2}{(m_1+m_2)}\, q \times \dot{q} \\
\\
J &= mq \times \dot{q}
\end{aligned}$$

This proves the claim in question 3.

At this point we're back to a problem you've already solved: a *single* particle in a central force. The only difference is that now q stands for the *relative* position and m stands for the *reduced* mass!

So, we instantly conclude that two bodies orbiting each other due to the force of gravity will *both* have an orbit that's either an ellipse, or a parabola, or a hyperbola... when viewed in the center-of-mass frame.

2 Poisson brackets

Let $\mathbb{R}^{2n}$ be the **phase space** of a particle in $\mathbb{R}^n$, with coordinates q_i, p_i $(1 \leq i \leq n)$. Let $C^\infty(\mathbb{R}^{2n})$ be the set of smooth real-valued functions on $\mathbb{R}^{2n}$, which becomes an commutative algebra using pointwise addition and multiplication of functions.

We define the **Poisson bracket** of functions $F, G \in C^\infty(\mathbb{R}^{2n})$ by:

$$\{F, G\} = \sum_{i=1}^{n} \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i} - \frac{\partial G}{\partial p_i}\frac{\partial F}{\partial q_i}.$$

4. Show that Poisson brackets make the vector space $C^\infty(R^{2n})$ into a **Lie algebra**. In other words, check the **antisymmetry** of the bracket:

$$\{F, G\} = -\{G, F\}$$

the **bilinearity** of the bracket:

$$\{F, \alpha G + \beta H\} = \alpha\{F, G\} + \beta\{F, H\}$$

$$\{\alpha F + \beta G, H\} = \alpha\{F, H\} + \beta\{G, H\}$$

and **Jacobi identity**:

$$\{F, \{G, H\}\} = \{\{F, G\}, H\} + \{G, \{F, H\}\}$$

for all $F, G, H \in C^\infty(\mathbb{R}^{2n})$ and $\alpha, \beta \in \mathbb{R}$.

(*Note the Jacobi identity resembles the product rule $d(GH) = (dG)H + GdH$, with bracketing by F playing the role of d. This is no accident!*)

Solution to 4:

Since,

$$\{F, G\} = \sum_{i=1}^{n} \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i} - \frac{\partial G}{\partial p_i}\frac{\partial F}{\partial q_i},$$

we can calculate the bracket when F and G are interchanged. We get:

$$\{G, F\} = \sum_{i=1}^{n} \frac{\partial G}{\partial p_i}\frac{\partial F}{\partial q_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}.$$

But then it becomes clear that:

$$-\{G, F\} = -\left[\sum_{i=1}^{n} \frac{\partial G}{\partial p_i}\frac{\partial F}{\partial q_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}\right] = \sum_{i=1}^{n} -\frac{\partial G}{\partial p_i}\frac{\partial F}{\partial q_i} + \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i} = \sum_{i=1}^{n} \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i} - \frac{\partial G}{\partial p_i}\frac{\partial F}{\partial q_i} = \{F, G\}$$

Proving the **antisymmetry** of the bracket, namely that:

$$\{F, G\} = -\{G, F\}.$$

Next, we investigate the **bilinearity**, writing $\{F,\, \alpha G + \beta H\}$ explicitly gives:

$$\begin{aligned}
\{F,\, \alpha\, G + \beta\, H\} &= \sum_{i=1}^{n} \frac{\partial F}{\partial p_i}\frac{\partial(\alpha\, G + \beta\, H)}{\partial q_i} - \frac{\partial(\alpha\, G + \beta\, H)}{\partial p_i}\frac{\partial F}{\partial q_i} \\
&= \sum_{i=1}^{n} \frac{\partial F}{\partial p_i}\left[\alpha\frac{\partial G}{\partial q_i} + \beta\frac{\partial H}{\partial q_i}\right] - \left[\alpha\frac{\partial G}{\partial p_i} + \beta\frac{\partial H}{\partial p_i}\right]\frac{\partial F}{\partial q_i} \\
&= \sum_{i=1}^{n} \alpha\frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i} + \beta\frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q_i} - \alpha\frac{\partial G}{\partial p_i}\frac{\partial F}{\partial q_i} - \beta\frac{\partial H}{\partial p_i}\frac{\partial F}{\partial q_i} \\
&= \sum_{i=1}^{n} \alpha\frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i} - \alpha\frac{\partial G}{\partial p_i}\frac{\partial F}{\partial q_i} + \beta\frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q_i} - \beta\frac{\partial H}{\partial p_i}\frac{\partial F}{\partial q_i} \\
&= \alpha\sum_{i=1}^{n} \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i} - \frac{\partial G}{\partial p_i}\frac{\partial F}{\partial q_i} + \beta\sum_{i=1}^{n} \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q_i} - \frac{\partial H}{\partial p_i}\frac{\partial F}{\partial q_i} \\
&= \alpha\,\{F,\, H\} + \beta\,\{G,\, H\}\}
\end{aligned}$$

For comlpeteness we must now check, $\{\alpha F + \beta G,\, H\}$, as well.

$$\begin{aligned}
\{\alpha F + \beta G,\, H\} &= \sum_{i=1}^{n} \frac{\partial(\alpha\, F + \beta\, G)}{\partial p_i}\frac{\partial H}{\partial q_i} - \frac{\partial H}{\partial p_i}\frac{\partial(\alpha\, F + \beta\, G)}{\partial q_i} \\
&= \sum_{i=1}^{n} \left[\alpha\frac{\partial F}{\partial p_i} + \beta\frac{\partial G}{\partial p_i}\right]\frac{\partial H}{\partial q_i} - \frac{\partial H}{\partial p_i}\left[\alpha\frac{\partial F}{\partial q_i} + \beta\frac{\partial G}{\partial q_i}\right] \\
&= \sum_{i=1}^{n} \alpha\frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q_i} + \beta\frac{\partial G}{\partial p_i}\frac{\partial H}{\partial q_i} - \alpha\frac{\partial H}{\partial p_i}\frac{\partial F}{\partial q_i} - \beta\frac{\partial H}{\partial p_i}\frac{\partial G}{\partial q_i} \\
&= \sum_{i=1}^{n} \alpha\frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q_i} - \alpha\frac{\partial H}{\partial q_i}\frac{\partial F}{\partial q_i} + \beta\frac{\partial G}{\partial p_i}\frac{\partial H}{\partial q_i} - \beta\frac{\partial H}{\partial q_i}\frac{\partial G}{\partial q_i} \\
&= \alpha\sum_{i=1}^{n} \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial F}{\partial q_i} + \beta\sum_{i=1}^{n} \frac{\partial G}{\partial p_i}\frac{\partial H}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial G}{\partial q_i} \\
&= \alpha\,\{F,\, H\} + \beta\,\{G,\, H\}\}
\end{aligned}$$

Thus the bracket fulfills the proptery of **bilinearity**. Lastly, we show that the **Jacobi Identity** holds.

$$\begin{aligned}
\{F,\, \{G,\, H\}\} &= \sum_{i=1}^{n} \frac{\partial F}{\partial p_i}\frac{\partial\{G,\, H\}}{\partial q_i} - \frac{\partial\{G,\, H\}}{\partial p_i}\frac{\partial F}{\partial q_i} \\
&= \sum_{i=1}^{n} \frac{\partial F}{\partial p_i}\frac{\partial}{\partial q_i}\left[\sum_{j=1}^{n} \frac{\partial G}{\partial p_j}\frac{\partial H}{\partial q_j} - \frac{\partial H}{\partial p_j}\frac{\partial G}{\partial q_j}\right] - \frac{\partial}{\partial p_i}\left[\sum_{j=1}^{n} \frac{\partial G}{\partial p_j}\frac{\partial H}{\partial q_j} - \frac{\partial H}{\partial p_j}\frac{\partial G}{\partial q_j}\right]\frac{\partial F}{\partial q_i} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} \left[\frac{\partial F}{\partial p_i}\frac{\partial}{\partial q_i}\left(\frac{\partial G}{\partial p_j}\frac{\partial H}{\partial q_j} - \frac{\partial H}{\partial p_j}\frac{\partial G}{\partial q_j}\right) - \frac{\partial}{\partial p_i}\left(\frac{\partial G}{\partial p_j}\frac{\partial H}{\partial q_j} - \frac{\partial H}{\partial p_j}\frac{\partial G}{\partial q_j}\right)\frac{\partial F}{\partial q_i}\right] \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial F}{\partial p_i}\left(\frac{\partial^2 G}{\partial p_j \partial q_i}\frac{\partial H}{\partial q_j} + \frac{\partial G}{\partial p_j}\frac{\partial^2 H}{\partial q_j \partial q_i} - \frac{\partial^2 H}{\partial p_j \partial q_i}\frac{\partial G}{\partial q_j} - \frac{\partial H}{\partial p_j}\frac{\partial^2 G}{\partial q_j \partial q_i}\right) \\
&\qquad - \left(\frac{\partial^2 G}{\partial p_j \partial p_i}\frac{\partial H}{\partial q_j} + \frac{\partial G}{\partial p_j}\frac{\partial^2 H}{\partial q_j \partial p_i} - \frac{\partial^2 H}{\partial p_j \partial p_i}\frac{\partial G}{\partial q_j} - \frac{\partial H}{\partial p_j}\frac{\partial^2 G}{\partial q_j \partial p_i}\right)\frac{\partial F}{\partial q_i}
\end{aligned}$$

$$
\begin{aligned}
= \quad & \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial F}{\partial p_i}\frac{\partial^2 G}{\partial p_j \partial q_i}\frac{\partial H}{\partial q_j} + \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial p_j}\frac{\partial^2 H}{\partial q_j \partial q_i} - \frac{\partial F}{\partial p_i}\frac{\partial^2 H}{\partial p_j \partial q_i}\frac{\partial G}{\partial q_j} - \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial p_j}\frac{\partial^2 G}{\partial q_j \partial q_i} \\
& \quad - \frac{\partial^2 G}{\partial p_j \partial p_i}\frac{\partial H}{\partial q_j}\frac{\partial F}{\partial q_i} - \frac{\partial G}{\partial p_j}\frac{\partial^2 H}{\partial q_j \partial p_i}\frac{\partial F}{\partial q_i} + \frac{\partial^2 H}{\partial p_j \partial p_i}\frac{\partial G}{\partial q_j}\frac{\partial F}{\partial q_i} + \frac{\partial H}{\partial p_j}\frac{\partial^2 G}{\partial q_j \partial p_i}\frac{\partial F}{\partial q_i} \\
= \quad & \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial F}{\partial p_i}\frac{\partial^2 G}{\partial p_j \partial q_i}\frac{\partial H}{\partial q_j} + \left(\frac{\partial^2 F}{\partial p_i \partial p_j}\frac{\partial G}{\partial q_i}\frac{\partial H}{\partial q_j} - \frac{\partial^2 F}{\partial p_i \partial p_j}\frac{\partial G}{\partial q_i}\frac{\partial H}{\partial q_j}\right) + \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial p_j}\frac{\partial^2 H}{\partial q_j \partial q_i} \\
& \quad - \frac{\partial F}{\partial p_i}\frac{\partial^2 H}{\partial p_j \partial q_i}\frac{\partial G}{\partial q_j} - \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial p_j}\frac{\partial^2 G}{\partial q_j \partial q_i} - \left(\frac{\partial^2 F}{\partial p_j \partial q_i}\frac{\partial G}{\partial p_i}\frac{\partial H}{\partial q_j} - \frac{\partial^2 F}{\partial p_j \partial q_i}\frac{\partial G}{\partial p_i}\frac{\partial H}{\partial q_j}\right) \\
& \quad - \frac{\partial^2 G}{\partial p_j \partial p_i}\frac{\partial H}{\partial q_j}\frac{\partial F}{\partial q_i} - \frac{\partial G}{\partial p_j}\frac{\partial^2 H}{\partial q_j \partial p_i}\frac{\partial F}{\partial q_i} - \left(\frac{\partial^2 F}{\partial p_i \partial q_j}\frac{\partial G}{\partial q_i}\frac{\partial H}{\partial p_j} - \frac{\partial^2 F}{\partial p_i \partial q_j}\frac{\partial G}{\partial q_i}\frac{\partial H}{\partial p_j}\right) \\
& \quad + \frac{\partial^2 H}{\partial p_j \partial p_i}\frac{\partial G}{\partial q_j}\frac{\partial F}{\partial q_i} + \frac{\partial H}{\partial p_j}\frac{\partial^2 G}{\partial q_j \partial p_i}\frac{\partial F}{\partial q_i} + \left(\frac{\partial^2 F}{\partial q_i \partial q_j}\frac{\partial G}{\partial p_i}\frac{\partial H}{\partial p_j} - \frac{\partial^2 F}{\partial q_i \partial q_j}\frac{\partial G}{\partial p_i}\frac{\partial H}{\partial p_j}\right) \\
= \quad & \sum_{i=1}^{n}\sum_{j=1}^{n} \left(\frac{\partial F}{\partial p_i}\frac{\partial^2 G}{\partial p_j \partial q_i}\frac{\partial H}{\partial q_j} + \frac{\partial^2 F}{\partial p_i \partial p_j}\frac{\partial G}{\partial q_i}\frac{\partial H}{\partial q_j}\right) - \frac{\partial^2 G}{\partial p_j \partial p_i}\frac{\partial H}{\partial q_j}\frac{\partial F}{\partial q_i} - \frac{\partial^2 F}{\partial p_j \partial q_i}\frac{\partial G}{\partial p_i}\frac{\partial H}{\partial q_j} \\
& \quad - \frac{\partial^2 F}{\partial p_i \partial q_j}\frac{\partial G}{\partial q_i}\frac{\partial H}{\partial p_j} - \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial p_j}\frac{\partial^2 G}{\partial q_j \partial q_i} + \frac{\partial H}{\partial p_j}\frac{\partial^2 G}{\partial q_j \partial p_i}\frac{\partial F}{\partial q_i} + \frac{\partial^2 F}{\partial q_i \partial q_j}\frac{\partial G}{\partial p_i}\frac{\partial H}{\partial p_j} \\
& \quad + \frac{\partial^2 F}{\partial p_i \partial q_j}\frac{\partial G}{\partial q_i}\frac{\partial H}{\partial p_j} + \frac{\partial^2 H}{\partial p_j \partial p_i}\frac{\partial G}{\partial q_j}\frac{\partial F}{\partial q_i} - \left(\frac{\partial F}{\partial p_i}\frac{\partial^2 H}{\partial p_j \partial q_i}\frac{\partial G}{\partial q_j} + \frac{\partial^2 F}{\partial p_i \partial p_j}\frac{\partial G}{\partial q_i}\frac{\partial H}{\partial q_j}\right) \\
& \quad + \frac{\partial^2 F}{\partial p_j \partial q_i}\frac{\partial G}{\partial p_i}\frac{\partial H}{\partial q_j} + \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial p_j}\frac{\partial^2 H}{\partial q_j \partial q_i} - \left(\frac{\partial G}{\partial p_j}\frac{\partial^2 H}{\partial q_j \partial p_i}\frac{\partial F}{\partial q_i} + \frac{\partial^2 F}{\partial q_i \partial q_j}\frac{\partial G}{\partial p_i}\frac{\partial H}{\partial p_j}\right) \\
= \quad & \sum_{i=1}^{n}\sum_{j=1}^{n} \left(\frac{\partial F}{\partial p_i}\frac{\partial^2 G}{\partial p_j \partial q_i} + \frac{\partial^2 F}{\partial p_i \partial p_j}\frac{\partial G}{\partial q_i}\right)\frac{\partial H}{\partial q_j} - \left(\frac{\partial^2 G}{\partial p_j \partial p_i}\frac{\partial F}{\partial q_i} + \frac{\partial^2 F}{\partial p_j \partial q_i}\frac{\partial G}{\partial p_i}\right)\frac{\partial H}{\partial q_j} \\
& \quad - \frac{\partial H}{\partial p_j}\left(\frac{\partial^2 F}{\partial p_i \partial q_j}\frac{\partial G}{\partial q_i} + \frac{\partial F}{\partial p_i}\frac{\partial^2 G}{\partial q_j \partial q_i}\right) + \frac{\partial H}{\partial p_j}\left(\frac{\partial^2 G}{\partial q_j \partial p_i}\frac{\partial F}{\partial q_i} + \frac{\partial^2 F}{\partial q_i \partial q_j}\frac{\partial G}{\partial p_i}\right) \\
& \quad + \frac{\partial G}{\partial p_j}\left(\frac{\partial^2 F}{\partial p_i \partial q_j}\frac{\partial H}{\partial q_i} + \frac{\partial F}{\partial p_i}\frac{\partial^2 H}{\partial q_j \partial q_i}\right) - \frac{\partial G}{\partial p_j}\left(\frac{\partial^2 H}{\partial q_j \partial p_i}\frac{\partial F}{\partial q_i} + \frac{\partial^2 F}{\partial q_j \partial q_i}\frac{\partial H}{\partial p_i}\right) \\
& \quad + \left(\frac{\partial^2 F}{\partial p_j \partial q_i}\frac{\partial H}{\partial p_i} + \frac{\partial^2 H}{\partial p_j \partial p_i}\frac{\partial F}{\partial q_i}\right)\frac{\partial G}{\partial q_j} - \left(\frac{\partial F}{\partial p_i}\frac{\partial^2 H}{\partial p_j \partial q_i} + \frac{\partial^2 F}{\partial p_j \partial p_i}\frac{\partial H}{\partial q_i}\right)\frac{\partial G}{\partial q_j} \\
= \quad & \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial}{\partial p_j}\left(\frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i} - \frac{\partial G}{\partial p_i}\frac{\partial F}{\partial q_i}\right)\frac{\partial H}{\partial q_j} - \frac{\partial H}{\partial p_j}\frac{\partial}{\partial q_j}\left(\frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i} - \frac{\partial G}{\partial p_i}\frac{\partial F}{\partial q_i}\right) \\
& \quad + \frac{\partial G}{\partial p_j}\frac{\partial}{\partial q_j}\left(\frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q_i} - \frac{\partial H}{\partial p_i}\frac{\partial F}{\partial q_i}\right) - \frac{\partial}{\partial p_j}\left(\frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q_i} - \frac{\partial H}{\partial p_i}\frac{\partial F}{\partial q_i}\right)\frac{\partial G}{\partial p_j} \\
= \quad & \sum_{j=1}^{n} \frac{\partial\{F,\, G\}}{\partial p_j}\frac{\partial H}{\partial q_j} - \frac{\partial H}{\partial p_j}\frac{\partial\{F,\, G\}}{\partial q_j} + \frac{\partial G}{\partial p_j}\frac{\partial\{F,\, H\}}{\partial q_j} - \frac{\partial\{F,\, H\}}{\partial p_j}\frac{\partial G}{\partial p_j} \\
= \quad & \{\{F,\, G\},\, H\} + \{G,\, \{F,\, H\}\}.
\end{aligned}
$$

`This completes the proof of the algebraic properties.`

5. Show that Poisson brackets and ordinary multiplication of functions make the vector space $C^\infty(\mathbb{R}^{2n})$ into a **Poisson algebra**. This is a Lie algebra that is also a commutative algebra, with the bracket $\{F, G\}$ and the product FG related by the **Leibniz identity**:

$$\{F, GH\} = \{F, G\}H + G\{F, H\}.$$

(*Again this identity resembles the product rule!*)

Solution to 5:

`Expanding the bracket of a product below leads to:`

$$\begin{aligned}
\{F,\, GH\} &= \sum_{i=1}^{n} \frac{\partial F}{\partial p_i}\frac{\partial (GH)}{\partial q_i} - \frac{\partial (GH)}{\partial p_i}\frac{\partial F}{\partial q_i} \\
&= \sum_{i=1}^{n} \frac{\partial F}{\partial p_i}\left[\frac{\partial G}{\partial q_i}H + G\frac{\partial H}{\partial q_i}\right] - \left[\frac{\partial G}{\partial p_i}H + G\frac{\partial H}{\partial p_i}\right]\frac{\partial F}{\partial q_i} \\
&= \sum_{i=1}^{n} \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}H + \frac{\partial F}{\partial p_i}G\frac{\partial H}{\partial q_i} - \frac{\partial G}{\partial p_i}H\frac{\partial F}{\partial q_i} - G\frac{\partial H}{\partial p_i}\frac{\partial F}{\partial q_i} \\
&= \sum_{i=1}^{n} \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}H - \frac{\partial G}{\partial p_i}\frac{\partial F}{\partial q_i}H + G\frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q_i} - G\frac{\partial H}{\partial p_i}\frac{\partial F}{\partial q_i} \\
&= \sum_{i=1}^{n} \left[\frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i} - \frac{\partial G}{\partial p_i}\frac{\partial F}{\partial q_i}\right]H + \sum_{i=1}^{n} G\left[\frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q_i} - \frac{\partial H}{\partial p_i}\frac{\partial F}{\partial q_i}\right]
\end{aligned}$$

`Arriving at:`

$$\{F, GH\} = \{F, G\}H + G\{F, H\},$$

`quod erat demonstrandum.`