Angular Momentum and Rotations

In this problem we will see that angular momentum generates rotations for a particle in \( \mathbb{R}^n \). We begin by recalling a bit about rotations. Let \( O(n) \) be the orthogonal group: the group of all linear transformations of \( \mathbb{R}^n \) that preserve distances. We can describe an element \( R \in O(n) \) as a real \( n \times n \) matrix that is orthogonal, meaning

\[ RR^* = R^* R = I \]

where \( R^* \) is the adjoint of the matrix \( R \) and \( I \) is the identity matrix.

We can define the exponential of any \( n \times n \) real matrix \( A \) to be the matrix defined by

\[ \exp(A) = \sum_{n=0}^{\infty} \frac{A^n}{n!} \]

(This series always converges.) Some easy calculations show that

\[ \exp((s + t)A) = \exp(sA) \exp(tA) \]

for all \( s, t \in \mathbb{R} \). Also, the entries of the matrix \( \exp(tA) \) are smooth functions of \( t \in \mathbb{R} \).

1. Suppose that \( A \) is skew-adjoint, meaning \( A^* = -A \). Show that \( \exp(tA) \in O(n) \) for all \( t \in \mathbb{R} \).

The group \( O(n) \) includes both rotations and reflections. In particular, \( O(n) \) consists of two connected components — the component where \( \det(R) = 1 \) and the component where \( \det(R) = -1 \). We define the rotation group or special orthogonal group \( SO(n) \) to be the subgroup consisting of all \( R \in O(n) \) with \( \det(R) = 1 \). This subgroup only includes rotations. A continuous curve can never go from one component to another. So, if \( A \) is skew-adjoint, \( \exp(tA) \) must actually lie in \( SO(n) \) for all \( t \).

We define \( so(n) \) to be the set of all skew-adjoint real \( n \times n \) matrices. This set \( so(n) \) is actually a Lie algebra, since it is a vector space closed under the bracket operation \([x, y] = xy - yx\). It is called the Lie algebra of the rotation group.

Now, let \( \mathbb{R}^{2n} \) be the phase space for a particle in \( \mathbb{R}^n \). A point \( (q, p) \in \mathbb{R}^{2n} \) describes the particle’s position \( q \in \mathbb{R}^n \) and momentum \( p \in \mathbb{R}^n \). The algebra of smooth real-valued functions \( C^\infty(\mathbb{R}^{2n}) \) becomes a Poisson algebra with

\[ \{F, G\} = \sum_{i=1}^{n} \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} - \frac{\partial G}{\partial p_i} \frac{\partial F}{\partial q_i}. \]

2. Given \( A \in so(n) \), let

\[ \phi: \mathbb{R} \times \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n} \]

be given by

\[ \phi(t, q, p) = (\exp(tA)q, \exp(tA)p). \]

Using the facts I’ve told you, show that \( \phi \) is a flow.
(For example, in 3 dimensions, this flow would rotate both the position and the momentum about some axis.)

3. Given $A \in \mathfrak{so}(n)$, define an observable $F \in C^\infty(\mathbb{R}^{2n})$ by

$$F(q,p) = \sum_{i,j=1}^{n} A_{ij}(q_i p_j - q_j p_i).$$

Show that some multiple of $F$ generates the flow $\phi$ defined above.

(I say ‘some multiple’ because you may need a factor of $\frac{1}{2}$ or a minus sign or something in front of $F$ to make this calculation work. I leave that to you!)

The moral: The observable that generates the flow $\phi$ is called \textbf{angular momentum in the $A$ direction}. But beware: $A$ is not a vector in $\mathbb{R}^n$! It’s a matrix in $\mathfrak{so}(n)$! For $n = 3$ we have an isomorphism

$$\mathfrak{so}(n) \cong \mathbb{R}^n$$

so we can talk about angular momentum in some direction $v \in \mathbb{R}^n$. But, this is not true in any other dimension (except $n = 0$)!

4. When $n = 3$, the observable

$$F(q,p) = q_1 p_2 - q_2 p_1$$

is usually called \textbf{angular momentum in the $z$ direction} and denoted $J_z$. What flow does this observable generate?