Classical Mechanics Homework
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John Baez homework by C.Pro

Angular Momentum and Rotations

In this problem we will see that angular momentum generates rotations for a particle in $\mathbb{R}^n$. We begin by recalling a bit about rotations. Let $\mathrm{O}(n)$ be the **orthogonal group**: the group of all linear transformations of $\mathbb{R}^n$ that preserve distances. We can describe an element $R \in \mathrm{O}(n)$ as a real $n \times n$ matrix that is **orthogonal**, meaning

$$OO^* = O^*O = I$$

where O^* is the adjoint of the matrix O and I is the identity matrix.

We can define the **exponential** of any $n \times n$ real matrix A to be the matrix defined by

$$\exp(A) = \sum_{n=0}^{\infty} \frac{A^n}{n!}$$

(This series always converges.) Some easy calculations show that

$$\exp((s+t)A) = \exp(sA)\exp(tA)$$

for all $s, t \in \mathbb{R}$. Also, the entries of the matrix $\exp(tA)$ are smooth functions of $t \in \mathbb{R}$.

1. Suppose that A is **skew-adjoint**, meaning $A^* = -A$, then $\exp(tA) \in \mathrm{O}(n)$ for all $t \in \mathbb{R}$.
   Proof: Using the fact for any $A, B \in M_n(\mathbb{R})$ we have $(A+B)^* = A^* + B^*$ and $(tAB)^* = t(B^*A^*)$, it follows that

$$\left(\sum_{n=0}^{k} \frac{A^n}{n!}\right)^* = \sum_{n=0}^{k} \frac{(A^*)^n}{n!}.$$

Since the exponential of a matrix converges for all $A \in M_n(\mathbb{R})$, the above implies $\exp(tA)^* = \exp(tA^*)$. Therefore, if A is skew-adjoint,

$$\begin{aligned}
\exp(tA)\exp(tA)^* &= \exp(t(A + A^*)) \\
&= \exp(0) \\
&= I,
\end{aligned}$$

The group $\mathrm{O}(n)$ includes both rotations and reflections. In particular, $\mathrm{O}(n)$ consists of two connected components — the component where $\det(R) = 1$ and the component where $\det(R) = -1$. We define the **rotation group** or **special orthogonal group** $\mathrm{SO}(n)$ to be the subgroup consisting of all $R \in \mathrm{O}(n)$ with $\det(R) = 1$. This subgroup only includes rotations. A continuous curve can never go from one component to another. So, if A is skew-adjoint, $\exp(tA)$ must actually lie in $\mathrm{SO}(n)$ for all t.

We define $\mathfrak{so}(n)$ to be the set of all skew-adjoint real $n \times n$ matrices. This set $\mathfrak{so}(n)$ is actually a Lie algebra, since it is a vector space closed under the bracket operation $[x, y] = xy - yx$. It is called the **Lie algebra of the rotation group**.

Now, let $\mathbb{R}^{2n}$ be the phase space for a particle in $\mathbb{R}^n$. A point $(q,p) \in \mathbb{R}^{2n}$ describes the particle's **position** $q \in \mathbb{R}^n$ and **momentum** $p \in \mathbb{R}^n$. The algebra of smooth real-valued functions $C^\infty(\mathbb{R}^{2n})$ becomes a Poisson algebra with

$$\{F,G\} = \sum_{i=1}^{n} \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i} - \frac{\partial G}{\partial p_i}\frac{\partial F}{\partial q_i}.$$

2. Given $A \in \mathfrak{so}(n)$, let

$$\phi\colon \mathbb{R} \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$$

be given by

$$\phi(t,q,p) = (\exp(tA)q, \exp(tA)p),$$

then ϕ is a flow.

Proof: Note that

$$\begin{aligned}\phi_0(q,p) &= (\exp(0)q, \exp(0)p)\\ &= (Iq, Ip)\\ &= (q,p)\end{aligned}$$

and

$$\begin{aligned}\phi_t(\phi_s(q,p)) &= \phi_t(\exp(sA)q, \exp(sA)p)\\ &= (\exp(tA)\exp(sA)q, \exp(tA)\exp(sA)p)\\ &= (\exp((t+s)A)q, \exp((t+s)Ap)\\ &= \phi_{t+s}(q,p).\end{aligned}$$

Since ϕ is clearly smooth, ϕ is a flow.

(For example, in 3 dimensions, this flow would rotate both the position and the momentum about some axis.)

3. Given $A \in \mathfrak{so}(n)$, define an observable $F \in C^\infty(\mathbb{R}^{2n}$ by

$$F(q,p) = \frac{1}{2}\sum_{i,j=1}^{n} A_{ij}(q_j p_i - q_i p_j),$$

then F generates the flow ϕ defined above.

Proof: For simplicity of notation let $\partial/\partial q$ and $\partial/\partial p$) represent the tangent vectors $(\partial/\partial q_1, \ldots, \partial/\partial q_n, 0, \ldots, 0)$ and $(0, \ldots, 0, \partial/\partial p_1, \ldots, \partial/\partial p_n)$, respectively. It then follows from the definition that $\{q, \cdot\} = -\partial/\partial p$ and $\{p, \cdot\} = \partial/\partial q$.

Note that $A_{ij} = -A_{ji}$, so

$$\begin{aligned}F(q,p) &= \frac{1}{2}\sum_{i,j=1}^{n} A_{ij}(q_j p_i - q_i p_j)\\ &= \frac{1}{2}\sum_{i,j=1}^{n} -A_{ji}q_j p_i - \sum_{i,j=1}^{n} A_{ij}q_i p_j)\\ &= -\sum_{i,j=1}^{n} A_{ij}q_i p_j\\ &= -q^* Ap.\end{aligned}$$

Therfore, by the Leibnitz rule we have

$$\begin{aligned} v_F &= \{-q^* A p, \cdot\} \\ &= -(\{p,\cdot\}^* q^*)A - Ap\{q,\cdot\} \\ &= -A^* q\frac{\partial}{\partial q} + Ap\frac{\partial}{\partial p} \\ &= Aq\frac{\partial}{\partial q} + Ap\frac{\partial}{\partial p}. \end{aligned}$$

Let ψ be the flow gernerated by v_F, and write $\psi_t(q,p) = (q(t), p(t))$. We must have

$$\begin{aligned} \dot{q}(t) &= Aq(t) \\ \dot{p}(t) &= Ap(t) \end{aligned}$$

and we see that $\psi(q,p) = (\exp(tA)q, \exp(tA)p)$ is a solution. By uniquness, it follows that $\psi = \phi$.

The moral: The observable that generates the flow ϕ is called **angular momentum in the** A **direction**. But beware: A is not a vector in $\mathbb{R}^n$! It's a matrix in $\mathfrak{so}(n)$! For $n = 3$ we have an isomorphism

$$\mathfrak{so}(n) \cong \mathbb{R}^n$$

so we can talk about angular momentum in some direction $v \in \mathbb{R}^n$. But, this is not true in any other dimension (except $n = 0$)!

4. When $n = 3$, the observable

$$F(q,p) = q_1 p_2 - q_2 p_1$$

is usually called **angular momentum in the** z **direction** and denoted J_z. What flow does this observable generate?

Solution: With a fixed basis $(\partial/\partial q_1, \ldots, \partial/\partial p_3)$ for the tangent space the vector field generated by F is:

$$\begin{aligned} v_F &= \{q_1 p_2 - q_2 p_1, \cdot\} \\ &= (-q_2, q_1, 0, -p_2, p_1, 0). \end{aligned}$$

Let $\phi_t(q,p) = (q(t), p(t))$ be the flow generated by v_F. When viewing $\mathbb{R}^3$ as $\mathbb{C} \times \mathbb{R}$, the above says:

$$\begin{aligned} \frac{d}{dt}(q_1(t) + iq_2(t)) = i(q_1(t) + iq_2(t)) \\ \frac{d}{dt}(p_1(t) + ip_2(t)) = i(q_1(t) + ip_2(t)) \\ \frac{d}{dt}q_3(t) = 0 \\ \frac{d}{dt}p_3(t) = 0, \end{aligned}$$

from which we can immediately conclude that

$$\begin{aligned} q_1(t) + iq_2(t) = e^{it}(q_1 + iq_2) \\ p_1(t) + ip_2(t) = e^{it}(p_1 + ip_2) \end{aligned}$$

and $q_3(t) = a$ and $p_3(t) = b$ are constants. In otherwords,

$$\phi_t(q_1 + iq_2, q_3, p_1 + ip_2, p_3) = (e^{it}(q_1 + iq_2), a, e^{it}(p_1 + ip_2), b).$$