Classical Mechanics, Lecture 10
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1 How Observables Generate Symmetries

Hamilton's equations are first-order differential equations. In the language of differential geometry, they are all about a certain vector field and the 'flow' it 'generates':

picture of manifold X with vector field and integral curve

A vector field v on a manifold X, we say a smooth function or **curve**

$$\gamma\colon \mathbb{R} \to X$$

is the **integral curve** of v through $x \in X$ if:

1. $\gamma(0) = x$
2. $\frac{d}{dt}\gamma(t) = v(\gamma(t)), \quad \forall t \in \mathbb{R}$

We say a vector field v is **integrable** if $\forall x \in X$ there exists an integral curve of v through x.

Example - $X = (0,1)$ and the vector field: $\frac{\partial}{\partial x}$. If we try to get the integral curve through $x \in (0,1)$ we get

$$\gamma(t) = x + t$$

but this is not in $(0,1)$ for t large! So, this vector field is not integrable.

Example - $X = \mathbb{R}$. This is secretly the same, but anyway: let

$$v = x^2 \frac{\partial}{\partial x}$$

Here our integral curve would satisfy:

$$\frac{d}{dt}\gamma(t) = \gamma(t)^2$$

$$\begin{aligned} \frac{dy}{dt} &= y^2 \\ \int \frac{dy}{y^2} &= \int dt \\ -\frac{1}{y} &= t + C \\ y &= -\frac{1}{t+C} \end{aligned}$$

i.e.,

$$\gamma(t) = -\frac{1}{t+C}$$

The problem is that this solution is not defined for all t — it blows up at $t = -C$. So, this vector field is also not integrable.

Suppose v is an integrable vector field on a manifold X. Then:

Theorem 1 *for every $x \in X$ the integral curve of v through x is unique.*

This let's us define a function:

$$\phi \colon \mathbb{R} \times X \to X$$

by

$$(t, x) \mapsto \phi(t, x) = \phi_t(x)$$

such that $\phi_t(x)$ is the integral curve of v through x.

Theorem 2 *$\phi \colon \mathbb{R} \times X \to X$ is smooth.*

Note also:

$$\phi_0(x) = x$$

and

$$\phi_s(\phi_t(x)) = \phi_{s+t}(x)$$

Mathematicians summarize these equations by saying "ϕ is an action of the group $(\mathbb{R}, +, 0)$ on X"; note they imply:

$$\phi_{-t}(x) = (\phi_t)^{-1}(x)$$

since

$$\phi_t \circ \phi_{-t} = \phi_0 = 1_X$$

So: for any $t \in \mathbb{R}$,

$$\phi_t \colon X \to X$$

is smooth (by Theorem) with a smooth inverse, ϕ_{-t}. A smooth map $f \colon X \to Y$ with smooth inverse is called a **diffeomorphism**.

Definition 3 *If $\phi \colon \mathbb{R} \times X \to X$ is a smooth map such that*

1. $\phi_0(x) = x$
2. $\phi_s(\phi_t(x)) = \phi_{s+t}(x)$

we call ϕ a **flow**.

We've seen that any integrable vector field v gives a flow ϕ: we call ϕ the flow **generated** by v. Conversely, any flow ϕ is generated by a unique (integrable) vector field v:

$$v(x) = \frac{d}{dt}\phi_t(x)|_{t=0}, \quad x \in X$$

Now suppose X is a Poisson manifold. If $H \in C^\infty(X)$ is any observable, thought of as the Hamiltonian, we get a vector field

$$\{H, \cdot\} \colon C^\infty(X) \to C^\infty(X)$$

also called v_H, the **Hamiltonian vector field generated by** H. If v_H is integrable, it generates a flow

$$\phi \colon \mathbb{R} \times X \to X$$

called **time evolution** or the **flow generated by** H. If our system is in the state $x \in X$ initially, then at time t it will be at $\phi_t(x) \in X$.