1 How Observables Generate Symmetries

Hamilton’s equations are first-order differential equations. In the language of differential geometry, they are all about a certain vector field and the ‘flow’ it ‘generates’:

A vector field \( v \) on a manifold \( X \), we say a smooth function or curve

\[ \gamma : \mathbb{R} \to X \]

is the integral curve of \( v \) through \( x \in X \) if:

1. \( \gamma(0) = x \)
2. \( \frac{d}{dt} \gamma(t) = v(\gamma(t)), \quad \forall t \in \mathbb{R} \)

We say a vector field \( v \) is integrable if \( \forall x \in X \) there exists an integral curve of \( v \) through \( x \).

**Example** - \( X = (0, 1) \) and the vector field: \( \frac{\partial}{\partial x} \). If we try to get the integral curve through \( x \in (0, 1) \) we get

\[ \gamma(t) = x + t \]

but this is not in \( (0, 1) \) for \( t \) large! So, this vector field is not integrable.

**Example** - \( X = \mathbb{R} \). This is secretly the same, but anyway: let

\[ v = x^2 \frac{\partial}{\partial x} \]

Here our integral curve would satisfy:

\[ \frac{d}{dt} \gamma(t) = (\gamma(t))^2 \]

\[ \frac{dy}{dt} = y^2 \]

\[ \int \frac{dy}{y^2} = \int dt \]

\[ -\frac{1}{y} = t + C \]

\[ y = \frac{1}{t + C} \]

i.e.,

\[ \gamma(t) = \frac{1}{t + C} \]

The problem is that this solution is not defined for all \( t \) — it blows up at \( t = -C \). So, this vector field is also not integrable.

Suppose \( v \) is an integrable vector field on a manifold \( X \). Then:
**Theorem 1** for every \( x \in X \) the integral curve of \( v \) through \( x \) is unique.

This lets us define a function:

\[ \phi: \mathbb{R} \times X \to X \]

by

\[ (t, x) \mapsto \phi(t, x) = \phi_t(x) \]

such that \( \phi_t(x) \) is the integral curve of \( v \) through \( x \).

**Theorem 2** \( \phi: \mathbb{R} \times X \to X \) is smooth.

Note also:

\[ \phi_0(x) = x \]

and

\[ \phi_s(\phi_t(x)) = \phi_{s+t}(x) \]

Mathematicians summarize these equations by saying “\( \phi \) is an action of the group \((\mathbb{R}, +, 0)\) on \( X \);” note they imply:

\[ \phi_{-t}(x) = (\phi_t)^{-1}(x) \]

since

\[ \phi_t \circ \phi_{-t} = \phi_0 = 1_X \]

So: for any \( t \in \mathbb{R} \),

\[ \phi_t: X \to X \]

is smooth (by Theorem) with a smooth inverse, \( \phi_{-t} \). A smooth map \( f: X \to Y \) with smooth inverse is called a **diffeomorphism**.

**Definition 3** If \( \phi: \mathbb{R} \times X \to X \) is a smooth map such that

1. \( \phi_0(x) = x \)
2. \( \phi_s(\phi_t(x)) = \phi_{s+t}(x) \)

we call \( \phi \) a **flow**.

We’ve seen that any integrable vector field \( v \) gives a flow \( \phi \): we call \( \phi \) the **flow generated** by \( v \). Conversely, any flow \( \phi \) is generated by a unique (integrable) vector field \( v \):

\[ v(x) = \frac{d}{dt} \phi_t(x)|_{t=0}, \quad x \in X \]

Now suppose \( X \) is a Poisson manifold. If \( H \in C^\infty(X) \) is any observable, thought of as the Hamiltonian, we get a vector field

\[ \{H, \cdot\}: C^\infty(X) \to C^\infty(X) \]

also called \( v_H \), the **Hamiltonian vector field generated** by \( H \). If \( v_H \) is integrable, it generates a flow

\[ \phi: \mathbb{R} \times X \to X \]

called **time evolution** or the **flow generated** by \( H \). If our system is in the state \( x \in X \) initially, then at time \( t \) it will be at \( \phi_t(x) \in X \).