1 Symmetries and Conserved Quantities

Let $X$ be a Poisson manifold. Let $F, G \in C^\infty(X)$. Suppose the vector fields

$$v_F = \{ F, \cdot \}$$
$$v_G = \{ G, \cdot \}$$

are integrable. Let

$$\phi : \mathbb{R} \times X \to X$$

be the flow generated by $F$:

$$\frac{d}{dt} \phi_t(x) = v_F(\phi_t(x)), \quad \forall t \in \mathbb{R}, x \in X$$

Let

$$\psi : \mathbb{R} \times X \to X$$

be the flow generated by $G$:

$$\frac{d}{dt} \psi_t(x) = v_G(\psi_t(x)), \quad \forall t \in \mathbb{R}, x \in X$$

If $F$ is “energy”, or “the Hamiltonian”, then $\phi_t : X \to X$ is called “time translation”. In this case we say $G$ is a conserved quantity (does not change as time passes) if:

$$G\phi_t = G, \quad \forall t \in \mathbb{R}$$

or

$$G(\phi_t(x)) = G(x), \quad \forall t \in \mathbb{R}, x \in X$$

If

$$F(\psi_t(x)) = F(x), \quad \forall t \in \mathbb{R}, x \in X$$

then we would say $G$ generates symmetries of $F$ - i.e. $F$ is constant along the integral curves of the flow generated by $G$.

**Theorem 1** $G$ generates symmetries of $F$ if and only if $F$ generates symmetries of $G$.

(If we think of $F$ as the “Hamiltonian”, we would say this as follows: $G$ generates symmetries of the Hamiltonian if and only if $G$ is conserved.)

**Proof**: $\forall t \in \mathbb{R}, x \in X,$

$$G$$ generates symmetries of $F \iff F(\psi_t(x)) = F(X)$$
$$\iff \frac{d}{dt} F(\psi_t(x)) = 0$$
$$\iff dF \left( \frac{d}{dt} \psi_t(x) \right) = 0$$
$$\iff dF(v_G(\psi_t(x))) = 0$$
\[ \begin{align*}
&\iff v_G(\psi_t(x))F = 0 \\
&\iff \{G, F\}(\psi_t(x)) = 0 \\
&\iff \{F, G\}(\psi_t(x)) = 0 \\
&\iff \{F, G\}(\phi_t(x)) = 0 \\
&\iff v_F(\phi_t(x))F = 0 \\
&\iff dG(v_F(\phi_t(x))) = 0 \\
&\iff dG(\frac{d}{dt}\phi_t(x)) = 0 \\
&\iff \frac{d}{dt}G(\phi_t(x)) = 0 \\
&\iff G(\phi_t(x)) = G(x) \\
&\iff F \text{ generates symmetries of } G
\end{align*} \]

Moral: the antisymmetry of the Poisson bracket is crucial!

**Theorem 2** $F$ generates symmetries of $F$.

(If $F$ is called the “Hamiltonian” this says: energy is conserved!)

*Proof:*

$F$ generates symmetries of $F$ \iff $F(\phi_t(x)) = F(x)$

\[ \iff \frac{d}{dt}F(\phi_t(x)) = 0 \]

\[ \iff dF \left( \frac{d}{dt}\phi_t(x) \right) = 0 \]

\[ \iff dF(v_F(\phi_t(x))) = 0 \]

\[ \iff v_F(F(\phi_t(x)) = 0 \]

\[ \iff \{F, F\}(\phi_t(x)) = 0 \]

but $\{F, F\} = -\{F, F\}$ so $\{F, F\} = 0$. Again, the antisymmetry of the Poisson bracket is crucial!

Given $F$ such that $v_F$ is integrable, let

\[ A = \{G \in C^\infty(X) | F \text{ generates symmetries of } G\} \]

\[ = \{G \in C^\infty(X) | G(\phi_t(x)) = G(x), \forall t, x\} \]

\[ = \{G \in C^\infty(X) | \{F, G\} = 0\} \]

If $F$ is called the “Hamiltonian”, elements of $A$ are called bf conserved quantities.

**Theorem 3** $A$ is a Poisson subalgebra of $C^\infty(X)$, i.e. it is closed under:

- linear combinations
- multiplication
- Poisson bracket
Proof:
Suppose $G, H \in A$.

1. $\alpha G + \beta H \in A, (\alpha, \beta \in \mathbb{R})$, since:
   \[
   \{ F, \alpha G + \beta H \} = \alpha \{ F, G \} + \beta \{ F, H \} = 0
   \]
   since $\{ \cdot, \cdot \}$ is bilinear.

2. $GH \in A$, since:
   \[
   \{ F, GH \} = \{ F, G \} H + G \{ F, H \} = 0
   \]
   since $\{ \cdot, \cdot \}$ satisfies the Leibniz law.

3. $\{ G, H \} \in A$, since:
   \[
   \{ F, \{ G, H \} \} = \{ \{ F, G \}, H \} + \{ G, \{ F, H \} \} = 0
   \]
   since $\{ \cdot, \cdot \}$ satisfies the Jacobi identity.

What we are doing is laying the groundwork for an axiomatic approach to classical mechanics. The key “axioms” would be:

1. observables form a commutative algebra
2. sufficiently nice observables generate “flows”
3. any observable generates a flow that leaves itself constant (generates symmetry of itself).
   (i.e., energy is always conserved!)

From axioms like this, we would like to derive the existence of a Poisson algebra of observables. 3 would give the antisymmetry.