Classical Mechanics, Lecture 12
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lecture by John Baez
notes by Alex Hoffnung

1 Symmetries and Conserved Quantities

Let X be a Poisson manifold. Let $F, G \in C^\infty(X)$. Suppose the vector fields

$$v_F = \{F, \cdot\}$$

$$v_G = \{G, \cdot\}$$

are integrable. Let

$$\phi: \mathbb{R} \times X \to X$$

be the flow generated by F:

$$\frac{d}{dt}\phi_t(x) = v_F(\phi_t(x)), \quad \forall t \in \mathbb{R}, x \in X$$

Let

$$\psi: \mathbb{R} \times X \to X$$

be the flow generated by G:

$$\frac{d}{dt}\psi_t(x) = v_G(\psi_t(x)), \quad \forall t \in \mathbb{R}, x \in X$$

If F is "energy", or "the Hamiltonian", then $\phi_t: X \to X$ is called "time translation". In this case we say G is a **conserved quantity** (does not change as time passes) if:

$$G\phi_t = G, \quad \forall t \in \mathbb{R}$$

or

$$G(\phi_t(x)) = G(x), \quad \forall t \in \mathbb{R}, x \in X$$

If

$$F(\psi_t(x)) = F(x), \quad \forall t \in \mathbb{R}, x \in X$$

then we would say G **generates symmetries** of F - i.e. F is constant along the integral curves of the flow generated by G.

Theorem 1 *G generates symmetries of F if and only if F generates symmetries of G.*

(If we think of F as the "Hamiltonian", we would say this as follows: G generates symmetries of the Hamiltonian if and only if G is conserved.)

Proof: $\forall t \in \mathbb{R}, x \in X$,

$$\begin{aligned}
G \text{ generates symmetries of } F \quad &\Leftrightarrow \quad F(\psi_t(x)) = F(X) \\
&\Leftrightarrow \quad \frac{d}{dt}F(\psi_t(x)) = 0 \\
&\Leftrightarrow \quad dF\left(\frac{d}{dt}\psi_t(x)\right) = 0 \\
&\Leftrightarrow \quad dF(v_G(\psi_t(x))) = 0
\end{aligned}$$

$$
\begin{aligned}
&\Leftrightarrow \quad v_G(\psi_t(x))F = 0 \\
&\Leftrightarrow \quad \{G, F\}(\psi_t(x)) = 0 \\
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&\Leftrightarrow \quad \{F, G\}(\phi_t(x)) = 0 \\
&\Leftrightarrow \quad v_F(\phi_t(x))F = 0 \\
&\Leftrightarrow \quad dG(v_F(\phi_t(x))) = 0 \\
&\Leftrightarrow \quad dG(\frac{d}{dt}\phi_t(x)) = 0 \\
&\Leftrightarrow \quad \frac{d}{dt}G(\phi_t(x)) = 0 \\
&\Leftrightarrow \quad G(\phi_t(x)) = G(x) \\
&\Leftrightarrow \quad F \text{ generates symmetries of } G
\end{aligned}
$$

Moral: the antisymmetry of the Poisson bracket is crucial!

Theorem 2 *F generates symmetries of F.*

(If F is called the "Hamiltonian" this says: energy is conserved!)

Proof:

$$
\begin{aligned}
F \text{ generates symmetries of } F \quad &\Leftrightarrow \quad F(\phi_t(x)) = F(x) \\
&\Leftrightarrow \quad \frac{d}{dt}F(\phi_t(x)) = 0 \\
&\Leftrightarrow \quad dF\left(\frac{d}{dt}\phi_t(x)\right) = 0 \\
&\Leftrightarrow \quad dF(v_F(\phi_t(x))) = 0 \\
&\Leftrightarrow \quad v_F(F\phi_t(x)) = 0 \\
&\Leftrightarrow \quad \{F, F\}(\phi_t(x)) = 0
\end{aligned}
$$

but $\{F, F\} = -\{F, F\}$ so $\{F, F\} = 0$. Again, the antisymmetry of the Poisson bracket is crucial!

Given F such that v_F is integrable, let

$$
\begin{aligned}
A &= \{G \in C^\infty(X) | F \text{ generates symmetries of } G\} \\
&= \{G \in C^\infty(X) | G(\phi_t(x)) = G(x), \forall t, x\} \\
&= \{G \in C^\infty(X) | \{F, G\} = 0\}
\end{aligned}
$$

If F is called the "Hamiltonian", elements of A are called bf conserved quantities.

Theorem 3 *A is a* **Poisson subalgebra** *of $C^\infty(X)$, i.e. it is closed under:*

- *linear combinations*
- *multiplication*
- *Poisson bracket*

Proof:
Suppose $G, H \in A$.

1. $\alpha G + \beta H \in A, (\alpha, \beta \in \mathbb{R})$,
   since:

   $$\{F, \alpha G + \beta H\} = \alpha\{F, G\} + \beta\{F, H\} = 0$$

   since $\{\cdot, \cdot\}$ is bilinear.

2. $GH \in A$, since:

   $$\{F, GH\} = \{F, G\}H + G\{F, H\} = 0$$

   since $\{\cdot, \cdot\}$ satisfies the Leibniz law.

3. $\{G, H\} \in A$, since:

   $$\{F, \{G, H\}\} = \{\{F, G\}, H\} + \{G, \{F, H\}\} = 0$$

   since $\{\cdot, \cdot\}$ satisfies the Jacobi identity.

What we are doing is laying the groundwork for an axiomatic approach to classical mechanics. The key "axioms" would be:

1. observables form a commutative algebra

2. sufficiently nice observables generate "flows"

3. any observable generates a flow that leaves itself constant (generates symmetriew of itself). (I.e., energy is always conserved!)

From axioms like this, we would like to derive the existence of a Poisson algebra of observables. 3 would give the antisymmetry.