Classical Mechanics, Lecture 18
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1 The Category of Classical Systems

A **category** is a bunch of things (**objects**) and processes (**morphisms**):

$$\bullet \longrightarrow \bullet$$

For us, the 'things' are physical systems in classical mechanics: Poisson manifolds. The 'processes' are certain maps between physical systems: Poisson maps.

Definition 1 *Given two Poisson manifolds X and Y, a map $\phi\colon X \to Y$ is* **Poisson** *if it is smooth and*

$$\phi^*\colon C^\infty(Y) \to C^\infty(X)$$

$$f \mapsto f\phi$$

preserves the Poisson brackets: given $f, g \in C^\infty(Y)$

$$\{\phi^*(f), \phi^*(g)\} = \phi^* \{f, g\}$$

In fact, we have seen lots of Poisson maps. Any physical process in classical mechanics gives a Poisson map.

Example: $X = T^*(\mathbb{R}^n)$ is the phase space for a particle in $\mathbb{R}^n$; the Galilei group $G(n+1)$ acts on X:

$$A\colon G(n+1) \times X \to X$$

and in fact for each $g \in G(n+1)$ the map $A(g)\colon X \to X$ is Poisson!

Example: X is any Poisson manifold, $H \in C^\infty(X)$. Then we get a vector field on X and if v_H is integrable we get a flow

$$\phi\colon \mathbb{R} \times X \to X$$

$$(t, x) \mapsto \phi_t(x)$$

with

$$\frac{d}{dt}\phi_t(x) = v_H(\phi_t(x))$$

Given this, each map $\phi_t\colon X \to X$ is Poisson!

Check: given $F, G \in C^\infty(X)$ we want:

$$\{\phi_t^* F, \phi_t^* G\} = \phi_t^* \{F, G\}$$

This is true for $t = 0$ since $\phi_0\colon X \to X$ is the identity, hence $\phi_0^* F = F$. So, it is enough to show:

$$\frac{d}{dt}\{\phi_t^* F, \phi_t^* G\} = \frac{d}{dt}\phi_t^* \{F, G\}$$

i.e.

$$\frac{d}{dt}\{\phi_t^* F, \phi_t^* G\}(x) = \frac{d}{dt}\phi_t^* \{F, G\}(x)$$

$$
\begin{aligned}
\frac{d}{dt}\phi_t^*\{F,G\}(x) &\overset{\text{def. of pullback}}{=} \frac{d}{dt}\{F,G\}(\phi_t(x)) \\
&\overset{\text{chain rule}}{=} d\{F,G\}\left(\frac{d}{dt}\phi_t(x)\right) \\
&\overset{\text{def. of }\phi_t}{=} d\{F,G\}\,v_H(\phi_t(x)) \\
&\overset{\text{def. of }v_H}{=} v_H\{F,G\}(\phi_t(x)) \\
&\overset{\text{Jacobi id.}}{=} \{H,\{F,G\}\}(\phi_t(x)) \\
&= (\{\{H,F\},G\}+\{F,\{H,G\}\})\,\phi_t(x)
\end{aligned}
$$

On the other hand:

$$
\begin{aligned}
\frac{d}{dt}\{\phi_t^*F,\phi_t^*G\}(x) &\overset{\text{bilin. of }\{\cdot,\cdot\}\text{ gives prod. rule}}{=} \left(\left\{\frac{d}{dt}\phi_t^*F,\phi_t^*G\right\}+\left\{\phi_t^*F,\frac{d}{dt}\phi_t^*G\right\}\right)(x) \\
&= \{\{H,\phi_t^*F\},\phi_t^*G\}+\{\phi_t^*F,\{H,\phi_t^*G\}\}(x)
\end{aligned}
$$

This agrees with the other side at $t=0$.
(Note: $H=\phi_t^*H$ since energy is conserved.)
So we have:

$$
\left.\frac{d}{dt}\phi_t^*\{F,G\}(x)\right|_{t=0} = \left.\frac{d}{dt}\{\phi_t^*F,\phi_t^*G\}(x)\right|_{t=0}
$$

for all x, so use $\phi_s(x)\in X$:

$$
\begin{aligned}
\left.\frac{d}{dt}\phi_t^*\{F,G\}\,\phi_s(x)\right|_{t=0} &= \left.\frac{d}{dt}\{\phi_t^*F,\phi_t^*G\}(\phi_s(x))\right|_{t=0} \\
\left.\frac{d}{dt}\{F,G\}\phi_{t+s}(x)\right|_{t=0} &= \\
\frac{d}{ds}\{F,G\}\phi_s(x) &=
\end{aligned}
$$

We would like:

$$
\frac{d}{ds}\{F,G\}\,\phi_s(x) = \frac{d}{ds}\{\phi_s^*F,\phi_s^*G\}
$$

Ugh!

So: we have a category of Poisson manifolds and Poisson maps and time evolution for any Hamiltonian is a Poisson map.

Definition 2 *A* **category** *consists of a collection of* **objects** *and for any pair of objects X and Y a set of* **morphisms** $f\colon X\to Y$ *such that given $f\colon X\to Y$ and $g\colon Y\to Z$ we have a morphism $gf\colon X\to Z$, such that:*

1. $(hg)f = h(gf)$
2. *each X has an* **identity morphism** $1_X\colon X\to X$ *such that:*

$$f1_X = f$$

$$1_X g = g$$