Classical Mechanics, Lecture 19

March 13, 2008
lecture by John Baez
notes by Alex Hoffnung

1 The Non-Cartesianness of Classical Mechanics

Last time we introduced a category Poiss, with

- Poisson manifolds X as objects
- Poisson maps $\phi\colon X \to Y$ as morphisms: smooth maps such that $\forall f, g \in C^\infty(Y)$

$$\phi^*\{f, g\} = \{\phi^* f, \phi^* g\}$$

  where $\phi^* f = f\phi$.

There can be various ways to "glom together" objects in a category - disjoint union, tensor products, Cartesian products, etc.. . .

For example: Set is the category with:

- sets X as objects
- functions $\phi\colon X \to Y$ as morphisms.

This has 'Cartesian product' $X \times Y$ as a way of glomming together sets. Here are the key properties of the Cartesian product, written so as to make sense in any category: we say the **product** $X \times Y$ is an object with morphisms

$$p_1\colon X \times Y \to X$$

$$p_2\colon X \times Y \to Y$$

such that: given any morphisms

$$f\colon Z \to X \text{ and } g\colon Z \to Y,$$

there exists a unique morphism $\langle f, g\rangle\colon Z \to X \times Y$ such that

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[d, dashed, "\langle f,g\rangle"] \arrow[dr, "g"] & \\ X & X \times Y \arrow[l, "p_1"] \arrow[r, "p_2"'] & Y \end{tikzcd}$$

commutes:

$$f = p_1\langle f, g\rangle$$

$$g = p_2\langle f, g\rangle.$$

In a 'cartesian' category, every pair of objects has a product.

Quantum theory uses not Poiss but the a category Hilb where:

- objects are Hilbert spaces
- morphisms are bounded linear operators,

Again, we use objects in this category to describe physical systems and morphisms to describe physical processes.

One reason quantum theory seems 'weird' to some people is that in this theory, we 'glom together' two physical systems using the tensor product of Hilbert spaces, which is *not* the 'product' in the sense just described!

I.e., given Hilbert spaces X and Y, we have this new Hilbert space $X \otimes Y$, but there are generally *not* any interesting morphisms

$$p_1 \colon X \otimes Y \to X$$

$$p_2 \colon X \otimes Y \to Y$$

For example, we use the vector $\psi \otimes \phi \in X \otimes Y$ to describe to describe a state of the system $X \otimes Y$ where the first subsystem is in the state ψ and the second subsystem is in the state ϕ. But, there are no linear operators as above that pick out these states:

$$p_1(\psi \otimes \phi) = \psi$$

$$p_2(\psi \otimes \phi) = \phi$$

for all $\psi \in X$, $\phi \in X$. Even more importantly, we can't find p_1, p_2 making $X \otimes Y$ into the product of X and Y: that is, operators such that for all $f \colon Z \to X$, $g \colon Z \to Y$, $\exists \langle f, g \rangle \colon Z \to X \otimes Y$ such that

$$\begin{tikzcd}
 & Z \arrow[dl, "f"'] \arrow[d, dashed, "\langle f,g \rangle"] \arrow[dr, "g"] & \\
X & X \otimes Y \arrow[l, "p_1"] \arrow[r, "p_2"'] & Y
\end{tikzcd}$$

commutes.

This has important consequences. For example, in a category with products, we can always "duplicate" a system: i.e. we have a morphism

$$\Delta_X \colon X \to X \times X.$$

We get this as follows:

$$\begin{tikzcd}
 & X \arrow[dl, "1_x"'] \arrow[d, dashed, "\Delta_x"] \arrow[dr, "1_x"] & \\
X & X \times X \arrow[l, "p_1"] \arrow[r, "p_2"'] & X
\end{tikzcd}$$

In the case of Set, we have

$$\Delta_X \colon X \to X \times X$$

$$x \mapsto (x, x).$$

But in Hilb we do not have any interesting linear operators

$$\Delta_X \colon X \to X \otimes X.$$

For example,

$$\psi \mapsto \psi \otimes \psi$$

is not linear. Wooters and Zurek proved a theorem making this issue precise: "you can not clone a quantum".

In fact, the right way of glomming together <u>classical</u> systems is also not the Cartesian product, but some kind of 'tensor product' of Poisson manifolds!

For example, if $X = T^*\mathbb{R}^n$ and $Y = T^*\mathbb{R}^m$ then

$$X \otimes Y \cong T^*\mathbb{R}^{n+m}$$

where all three have their usual Poisson brackets. As <u>manifolds</u>

$$T^*\mathbb{R}^{n+m} \cong T^*T^n \times T^*\mathbb{R}^m$$

$$(q, q', p, p') \mapsto ((q, p), (q', p'))$$

and this is a product in the category of manifolds and smooth maps. *But, it is not a product in the category of Poisson manifolds!*

I believe the non-Cartesian nature of this product means there's no classical machine that can 'duplicate' states of a classical system:

picture of classical machine where you feed a system intothe hamper and two identical copies come out the bottom

But, strangely, this issue has been studied less than in the quantum case!