Classical Mechanics, Lecture 3

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lecture by John Baez

notes by Alex Hoffnung

1 Conservation of Energy

Today we will talk about what conservation of energy is good for — how it can help us solve problems in classical mechanics. If we have a particle $q\colon \mathbb{R} \to \mathbb{R}^n$ satisfying $F = ma$ where F is **conservative**:

$$F(t) \;=\; -\nabla V(q(t))$$

where $V\colon \mathbb{R}^n \to \mathbb{R}$ is called the **potential**, then energy is conserved. Let

$$E(t) \;=\; \frac{1}{2}m\dot{q}(t)^2 \;+\; V(q(t)).$$

Then

$$\begin{aligned}
\frac{d}{dt}E(t) &= m\dot{q}(t)\cdot\ddot{q}(t) + \nabla V(q(t))\cdot\dot{q}(t) \\
&= F(q(t))\cdot\dot{q}(t) + \nabla V(q(t))\cdot\dot{q}(t) \\
&= 0
\end{aligned}$$

What good is this? It helps understand the motion of the particle: for any solution of Newton's second law

$$\frac{1}{2}m\dot{q}(t)^2 \;+\; V(q(t)) \;+\; E$$

$$||\,\dot{q}(t)\,|| \;=\; \sqrt{\frac{2}{m}(E - V(q(t)))}$$

so we know the particle's speed given its position. This is especially powerful for a particle on the line (n=1).

Example: A particle on a line. In this case, suppose the force depends only on position:

$$F(t) \;=\; f(q(t))$$

for $f\colon \mathbb{R} \to \mathbb{R}$. Then automatically F is conservative:

$$\begin{aligned}
f &= -\nabla V \\
&= -\frac{dV}{dx}
\end{aligned}$$

where

$$V(x) \;=\; \int_{x_0}^{x} f(s)ds.$$

Note: we can add any constant to V. Also: the fact that any f is $-\nabla V$ for some V is special to 1 dimension.

So we have:

$$|\,\dot{q}(t)\,| \;=\; \sqrt{\frac{2}{m}(E - V(q(t)))}$$

so

$$\dot{q}(t) = \pm\sqrt{\frac{2}{m}(E - V(q(t)))}$$

For example:

graph of some function $V(x)$ on plane with chosen energy value $V = E$

The particle's position, say x, must have

$$V(x) \leq E.$$

This is called the **classically allowed** region - in our example, $[x_0, x_1]$. The set of $x \in \mathbb{R}$ where $V(x) > E$ is the **classically forbidden**.
A particle at a local maximum can go one of the two possible directions. If the potential increases all the way up to $V = E$, the particle stops for moment and then Newton's second law demands that the particle goes back down the graph. In our example the particle must oscillate between x_0 and x_1, moving faster where V is smaller.

Example: A particle in $\mathbb{R}^3$ in a central force.

picture of a central force field

A **central force** depends only on position, so it's given by $f\colon \mathbb{R}^3 - \{0\} \to \mathbb{R}^3$, but where f is spherically symmetric:

$$f(x) = \phi(\| x \|)\frac{x}{\| x \|}$$

where $\phi\colon [0,\infty) \to \mathbb{R}$. (We'll worry about the origin in $\mathbb{R}^3$ when necessary.) We'll write

$$\| q(t) \| = r(t)$$

$$\frac{q(t)}{\| q(t) \|} = \hat{r}(t)$$

so Newton's 2nd law says

$$m\ddot{q}(t) = \phi(r(t))\hat{r}(t).$$

Kepler started thinking about planetary motion - this is motion in a central force

$$\phi(r) = -\frac{k}{r^2}$$

He noted that planets sweep out equal area in equal time:

picture of planet going around sun with area from t_0 to $t_0 + \Delta t$ and from t_1 to $t_1 + \Delta t$

This is secretly "conservation of angular momentum". This will let us understand motion in any central force.
First, a central force is automatically conservative: if

$$f(x) = \phi(\| x \|)\hat{x}, \qquad (\hat{x} = \frac{x}{\| x \|})$$

then

$$f(x) = -\nabla V(x)$$

where

$$V(x) = v(\| x \|)$$

for some $v\colon(0,\infty)\to\mathbb{R}$, namely any v with $v'=-\phi$, e.g.:

$$v(r) \;=\; -\int_{r_0}^{r_1}\phi(s)ds.$$

So we have conservation of energy

$$E(t) \;=\; \frac{1}{2}m\dot{q}(t)^2 \;=\; v(r(t))$$

(where $r(t) =\|\ q(t)\ \|$) is **constant**. But we also have conservation of **angular momentum** $J\colon\mathbb{R}\to\mathbb{R}^3$ given by

$$J(t) \;=\; q(t)\times p(t).$$

Why is this conserved?

$$\begin{aligned}\frac{d}{dt}J(t) &= \dot{q}(t)\times p(t)+q(t)\times\dot{p}(t)\\ &= \dot{q}(t)\times m\dot{q}(t)+q(t)\times F(t)\\ &= 0+q(t)\times f(q(t))\\ &= q(t)\times\phi(r(t))\hat{q}(t)\\ &= 0.\end{aligned}$$

In general: angular momentum is constant when the force points directly towards or away from the origin.

For a particle in a central force, $J(t)=mq(t)\times\dot{q}(t)$ is constant so $q(t)$ and $\dot{q}(t)$ must lie in some fixed plane, independent of t. So, choose coordinates so that it's the xy plane. So now we have a particle in $\mathbb{R}^2$. Let's describe its position using polar coordinates $r(t),\theta(t)$. In these coordinates:

$$E(t) \;=\; \frac{1}{2}m(\dot{r}(t)^2+r^2\dot{\theta}(t)^2) \;+\; v(r(t))$$

and $J(t)$ is pointing in the z-direction and proportional to

$$j(t) \;=\; mr(t)^2\dot{\theta}(t)$$

$E(t)$ and $j(t)$ are both consnstant - E and j. So:

$$\begin{aligned} j &= mr^2\dot{\theta}\\ \dot{\theta} &= \frac{j}{mr^2}\end{aligned}$$

So

$$E \;=\; \frac{1}{2}m(\dot{r}^2 \;+\; \frac{j^2}{m^2r^2}) \;+\; v(r(t)).$$

This is isomorphic to a particle on $(0,\infty)$ with position $r(t)$ and velocity $\dot{r}(t)$ and energy

$$E \;=\; \frac{1}{2}m\dot{r}(t)^2 \;+\; V_{eff}(r(t))$$

where the **effective potential** is

$$V_{eff}(r) \;=\; v(r) \;+\; \frac{1}{2}\frac{j^2}{mr^2}.$$

picture of typical potential and effective potential

So the "effective" force is the force due to V plus a **centrifugal** force due to $\frac{j^2}{mr^2}$.