

1 Conservation of Energy

Today we will talk about what conservation of energy is good for — how it can help us solve problems in classical mechanics. If we have a particle $q: \mathbb{R} \to \mathbb{R}^n$ satisfying $F = m\ddot{q}$ where $F$ is conservative:

$$F(t) = -\nabla V(q(t))$$

where $V: \mathbb{R}^n \to \mathbb{R}$ is called the potential, then energy is conserved. Let

$$E(t) = \frac{1}{2}m\dot{q}(t)^2 + V(q(t)).$$

Then

$$\frac{d}{dt}E(t) = m\dot{q}(t) \cdot \ddot{q}(t) + \nabla V(q(t)) \cdot \dot{q}(t) = F(q(t)) \cdot \dot{q}(t) + \nabla V(q(t)) \cdot \dot{q}(t) = 0$$

What good is this? It helps understand the motion of the particle: for any solution of Newton’s second law

$$\frac{1}{2}m\dot{q}(t)^2 + V(q(t)) + E$$

$$\| \dot{q}(t) \| = \sqrt{\frac{2}{m}(E - V(q(t)))}$$

so we know the particle’s speed given its position. This is especially powerful for a particle on the line ($n=1$).

Example: A particle on a line. In this case, suppose the force depends only on position:

$$F(t) = f(q(t))$$

for $f: \mathbb{R} \to \mathbb{R}$. Then automatically $F$ is conservative:

$$f = -\nabla V = -\frac{dV}{dx}$$

where

$$V(x) = \int_{x_0}^x f(s)ds.$$ 

Note: we can add any constant to $V$. Also: the fact that any $f$ is $-\nabla V$ for some $V$ is special to 1 dimension.

So we have:

$$| \dot{q}(t) | = \sqrt{\frac{2}{m}(E - V(q(t)))}$$
\[ \dot{q}(t) = \pm \sqrt{\frac{2}{m}(E - V(q(t)))} \]

For example:

graph of some function \( V(x) \) on plane with chosen energy value \( V = E \)

The particle’s position, say \( x \), must have

\[ V(x) \leq E. \]

This is called the classically allowed region - in our example, \([x_0, x_1]\). The set of \( x \in \mathbb{R} \) where \( V(x) > E \) is the classically forbidden.

A particle at a local maximum can go one of the two possible directions. If the potential increases all the way up to \( V = E \), the particle stops for moment and then Newton’s second law demands that the particle goes back down the graph. In our example the particle must oscillate between \( x_0 \) and \( x_1 \), moving faster where \( V \) is smaller.

Example: A particle in \( \mathbb{R}^3 \) in a central force.

\[ \text{picture of a central force field} \]

A central force depends only on position, so it’s given by \( f: \mathbb{R}^3 - \{0\} \rightarrow \mathbb{R}^3 \), but where \( f \) is spherically symmetric:

\[ f(x) = \phi(||x||) \frac{x}{||x||} \]

where \( \phi: [0, \infty) \rightarrow \mathbb{R} \). (We’ll worry about the origin in \( \mathbb{R}^3 \) when necessary.) We’ll write

\[ ||q(t)|| = r(t) \]
\[ \frac{q(t)}{||q(t)||} = \hat{r}(t) \]

so Newton’s 2nd law says

\[ m\ddot{q}(t) = \phi(r(t))\hat{r}(t). \]

Kepler started thinking about planetary motion - this is motion in a central force

\[ \phi(r) = -\frac{k}{r^2} \]

He noted that planets sweep out equal area in equal time:

\[ \text{picture of planet going around sun with area from } t_0 \text{ to } t_0 + \Delta t \text{ and from } t_1 \text{ to } t_1 + \Delta t \]

This is secretly “conservation of angular momentum”. This will let us understand motion in any central force.

First, a central force is automatically conservative: if

\[ f(x) = \phi(||x||)\hat{x}, \quad (\hat{x} = \frac{x}{||x||}) \]

then

\[ f(x) = -\nabla V(x) \]

where

\[ V(x) = v(||x||) \]
for some \( v : (0, \infty) \to \mathbb{R} \), namely any \( v \) with \( v' = -\phi \), e.g.:

\[
v(r) = -\int_{r_0}^{r_1} \phi(s) ds.
\]

So we have conservation of energy

\[
E(t) = \frac{1}{2} m \dot{q}(t)^2 = v(r(t))
\]

(where \( r(t) = || q(t) || \) is constant. But we also have conservation of angular momentum \( J : \mathbb{R} \to \mathbb{R}^3 \) given by

\[
J(t) = q(t) \times p(t).
\]

Why is this conserved?

\[
\frac{d}{dt} J(t) = \dot{q}(t) \times p(t) + q(t) \times \dot{p}(t)
\]

\[
= \dot{q}(t) \times m\dot{q}(t) + q(t) \times F(t)
\]

\[
= 0 + q(t) \times f(q(t))
\]

\[
= q(t) \times \phi(r(t)) \dot{q}(t)
\]

\[
= 0.
\]

In general: angular momentum is constant when the force points directly towards or away from the origin.

For a particle in a central force, \( J(t) = mq(t) \times \dot{q}(t) \) is constant so \( q(t) \) and \( \dot{q}(t) \) must lie in some fixed plane, independent of \( t \). So, choose coordinates so that it’s the \( xy \) plane. So now we have a particle in \( \mathbb{R}^2 \). Let’s describe its position using polar coordinates \( r(t), \theta(t) \). In these coordinates:

\[
E(t) = \frac{1}{2} m (\dot{r}(t)^2 + r^2 \dot{\theta}(t)^2) + v(r(t))
\]

and \( J(t) \) is pointing in the \( z \)-direction and proportional to

\[
j(t) = mr(t)^2 \dot{\theta}(t)
\]

\( E(t) \) and \( j(t) \) are both consntant - \( E \) and \( j \). So:

\[
j = mr^2 \dot{\theta}
\]

\[
\dot{\theta} = \frac{j}{mr^2}
\]

So

\[
E = \frac{1}{2} m (\dot{r}^2 + \frac{j^2}{mr^2}) + v(r(t)).
\]

This is isomorphic to a particle on \((0, \infty)\) with position \( r(t) \) and velocity \( \dot{r}(t) \) and energy

\[
E = \frac{1}{2} m \dot{r}(t)^2 + V_{eff}(r(t))
\]

where the effective potential is

\[
V_{eff}(r) = v(r) + \frac{1}{2} \frac{j^2}{mr^2}.
\]

picture of typical potential and effective potential

So the “effective” force is the force due to \( V \) plus a centrifugal force due to \( \frac{j^2}{mr^2} \).