1 Many particles in $\mathbb{R}^n$

Today we are going to talk about conservation of momentum. This is a modern name for Newton’s 3rd law. We haven’t discussed this law yet. It says:

All forces occur in pairs and these two forces are equal in magnitude and opposite in direction.

To understand this idea, we must consider systems with more than one particle — say $n$ particles in $\mathbb{R}^3$ with positions $q_i: \mathbb{R} \to \mathbb{R}^3$, $(i = 1, \ldots, n)$. Suppose the $j^{th}$ particle exerts some force on the $i^{th}$ particle $(i \neq j)$, $F_{ij}(t)$. Newton’s 3rd law says

$$F_{ij}(t) = - F_{ji}(t).$$

We will go further and assume our forces are central:

(to get conservation of angular momentum). More precisely:

$$F_{ij}(t) = f_{ij}(|q_i(t) - q_j(t)|) \frac{q_i(t) - q_j(t)}{||q_i(t) - q_j(t)||}$$

where $f_{ij}: (0, \infty) \to \mathbb{R}$. (Note the force is undefined when the particles collide!) Then Newton’s 3rd law says

$$f_{ji} = f_{ji}, \quad (i \neq j).$$

As for a single particle, we can define potentials $V_{ij}: (0, \infty) \to \mathbb{R}$ with

$$V'_{ij} = - f_{ij}.$$

For example:

$$V_{ij}(r) = - \int_c^r f_{ij}(s) ds$$

for some fixed $c$. Note:

$$V_{ij} = V_{ji}$$

so “the potential energy of the $i^{th}$ particle due to the $j^{th}$ particle equals the potential energy of the $j^{th}$ particle due to the $i^{th}$ particle.” We define the total potential energy of the $i^{th}$ particle:

$$V_i(t) = \sum_{j \neq i} V_{ij}(|q_i(t) - q_j(t)|)$$

and the total potential energy

$$V(t) = \sum_{i=1}^n V_i(t)$$
Similarly, we have the total kinetic energy

\[ T(t) = \sum_{i=1}^{n} \frac{1}{2} m_i \dot{q}_i(t)^2. \]

These add up to give the total energy

\[ E(t) = T(t) + V(t) \]

This will be conserved if the particles move according to Newton’s 2nd law:

\[ F_i(t) = m_i \ddot{q}_i(t), \quad (i = 1, \ldots, n) \]

where \( m > 0 \) is the mass of the \( i \)th particle and the force on the \( i \)th particle is

\[ F_i(t) = \sum_{j \neq i} F_{ij}(t) \]

where \( F_{ij} \) is given as before.

Homework 1: Show that if Newton’s 2nd law \( (F_i(t) = m_i \ddot{q}_i(t)) \) holds then energy is conserved:

\[ \frac{d}{dt} E(t) = 0. \]

More interestingly, Newton’s 3rd law \( (F_{ij} = F_{ji}) \) gives conservation of momentum. The momentum of the \( i \)th particle is:

\[ p_i(t) = m_i \dot{q}_i(t) \]

and the total momentum is:

\[ p(t) = \sum_{i=1}^{n} p_i(t). \]

This is conserved:

\[ \frac{d}{dt} p(t) = \sum_{i=1}^{n} \frac{d}{dt} p_i(t) \]

\[ = \sum_{i=1}^{n} m_i \dot{q}_i(t) \]

\[ = \sum_{i=1}^{n} F_i(t) \]

\[ = \sum_{i=1}^{n} \sum_{j \neq i} F_{ij}(t) \]

\[ = 0 \]

since \( F_{ij}(t) = -F_{ji}(t) \). We also have a third conservation law: conservation of angular momentum. The angular momentum of the \( i \)th particle is

\[ J_i(t) = m_i q_i(t) \times \dot{q}_i(t). \]

(If the particle and its velocity are in the xy plane then \( J_i(t) \) points along the z direction and its \( z \) component is \( m r(t)^2 \theta(t) \).) The total angular momentum is:

\[ J(t) = \sum_{i=1}^{n} J_i(t) \]
Homework 2: Show that \( \frac{d}{dt} J(t) = 0 \) using Newton’s 2\(^{nd}\) law and

\[
F_{ij}(t) = f_{ij}(\|q_i(t) - q_j(t)\|) \frac{q_i(t) - q_j(t)}{\|q_i(t) - q_j(t)\|}
\]

where \( f_{ij} = f_{ji} \). Eventually we will use all these conserved quantities to solve the “2-body problem” - two particles interacting via a central force. When we get to the 3-body problem, things get more complicated.

Question: consider the gravitational n-body problem:

\[
V_{ij}(r) = -\frac{Gm_i m_j}{r}
\]

where \( G \) is Newton’s constant, so

\[
f_{ij}(r) = \frac{Gm_i m_j}{r^2}.
\]

Given initial data \( q_i(0), \dot{q}_i(0) \) does there exist a solution of Newton’s 2\(^{nd}\) law:

\[
m\ddot{q}_i(t) = \sum_{j \neq i} -\frac{Gm_i m_j}{\|q_i(t) - q_j(t)\|^2} \frac{q_i(t) - q_j(t)}{\|q_i(t) - q_j(t)\|}
\]

for all values of \( t \in \mathbb{R} \)? Depending on initial data, we may have collision singularities where \( q_i(t) = q_j(t), (i \neq j) \), and then we can worry if we can continue the solution beyond the singularities. But: no, it is not true that we have a solution for all \( q_i(0), \dot{q}_i(0) \). So:

1. Is there a solution for almost everywhere choice of initial data?
   Answer is yes for \( n = 2, 3, 4, \ldots \) but open for large \( n \).

2. If there is no collision singularity, then is there a solution for all \( t \)?
   Not always! There is a counterexample for \( n = 5 \):

   ![picture of 5-body counterexample]

   Nobody knows of these non-collision singularities arise from a set of initial data of measure zero (if \( n \geq 5 \)).