Classical Mechanics, Lecture 4

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1 Many particles in $\mathbb{R}^n$

Today we are going to talk about **conservation of momentum**. This is a modern name for Newton's 3^{rd} law. We haven't discussed this law yet. It says:

All forces occur in pairs and these two forces are equal in magnitude and opposite in direction.

$$picture\ of\ Earth\ and\ apple\ with\ forces\ acting\ on\ eachother$$

To understand this idea, we must consider systems with more than one particle — say n particles in $\mathbb{R}^3$ with positions $q_i\colon \mathbb{R} \to \mathbb{R}^3$, $(i = 1, \ldots, n)$. Suppose the j^{th} particle exerts some force on the i^{th} particle $(i \neq j)$, $F_{ij}(t)$. Newton's 3^{rd} law says

$$F_{ij}(t) \;=\; -F_{ji}(t).$$

We will go further and assume our forces are central:

$$picture\ of\ arrows\ pointing\ along\ same\ line\ notalong\ distinct\ lines$$

(to get conservation of angular momentum). More precisely:

$$F_{ij}(t) \;=\; f_{ij}(|\, q_i(t) - q_j(t)\,|)\frac{q_i(t) - q_j(t)}{||\, q_i(t) - q_j(t)\,||)}$$

where $f_{ij}\colon (0, \infty) \to \mathbb{R}$. (Note the force is undefined when the particles collide!) Then Newton's 3^{rd} law says

$$f_{ji} \;=\; f_{ji}, \qquad (i \neq j).$$

As for a single particle, we can define potentials $V_{ij}\colon (0, \infty) \to \mathbb{R}$ with

$$V'_{ij} \;=\; -f_{ij}.$$

For example:

$$V_{ij}(r) \;=\; -\int_c^r f_{ij}(s)ds$$

for some fixed c. Note:

$$V_{ij} \;=\; V_{ji}$$

so "the potential energy of the i^{th} particle due to the j^{th} particle equals the potential energy of the j^{th} particle due to the i^{th} particle." We define the **total potential energy of the i^{th} particle**:

$$V_i(t) \;=\; \sum_{j \neq i} V_{ij}(|\, q_i(t) - q_j(t)\,|)$$

and the **total potential energy**

$$V(t) \;=\; \sum_{i=1}^{n} V_i(t)$$

Similarly, we have the **total kinetic energy**

$$T(t) \;=\; \sum_{i=1}^{n} \frac{1}{2} m_i \dot{q}_i(t)^2.$$

These add up to give the **total energy**

$$E(t) \;=\; T(t) \;+\; V(t)$$

This will be conserved if the particles move according to Newton's 2^{nd} law:

$$F_i(t) \;=\; m_i \ddot{q}_i(t), \qquad (i = 1, \cdot, n)$$

where $m > 0$ is the **mass** of the i^{th} particle and the **force** on the i^{th} particle is

$$F_i(t) \;=\; \sum_{j \neq i} F_{ij}(t)$$

where F_{ij} is given as before.

Homework 1: Show that if Newton's 2^{nd} law ($F_i(t) = m_i \ddot{q}_i(t)$) holds then energy is conserved:

$$\frac{d}{dt} E(t) \;=\; 0.$$

More interestingly, Newton's 3^{rd} law ($F_{ij} = F_{ji}$) gives conservation of momentum. The **momentum** of the i^{th} particle is:

$$p_i(t) \;=\; m_i \dot{q}_i(t)$$

and the **total momentum** is:

$$p(t) \;=\; \sum_{i=1}^{n} p_i(t).$$

This is conserved:

$$\begin{aligned}
\frac{d}{dt} p(t) \;&=\; \sum_{i=1}^{n} \frac{d}{dt} p_i(t) && (1)\\
&=\; \sum_{i=1}^{n} m_i \ddot{q}_i(t) && (2)\\
&=\; \sum_{i=1}^{n} F_i(t) && (3)\\
&=\; \sum_{i=1}^{n} \sum_{j \neq i} F_{ij}(t) && (4)\\
&=\; 0 && (5)
\end{aligned}$$

since $F_{ij}(t) \;=\; -F_{ji}(t)$. We also have a third conservation law: conservation of angular momentum. The **angular momentum** of the i^{th} particle is

$$J_i(t) \;=\; m_i q_i(t) \times \dot{q}_i(t).$$

(If the particle and its velocity are in the xy plane then $J_i(t)$ points along the z direction and its z component is $mr(t)^2 \dot{\theta}(t)$.) The **total angular momentum** is:

$$J(t) \;=\; \sum_{i=1}^{n} J_i(t)$$

Homework 2: Show that $\frac{d}{dt}J(t) = 0$ using Newton's 2^{nd} law and

$$F_{ij}(t) = f_{ij}(| q_i(t) - q_j(t) |)\frac{q_i(t) - q_j(t)}{| q_i(t) - q_j(t) |}$$

where $f_{ij} = f_{ji}$. Eventually we will use all these conserved quantities to solve the "2-body problem" - two particles interacting via a central force. When we get to the 3-body problem, things get more complicated.

Question: consider the gravitational n-body problem:

$$V_{ij}(r) = -\frac{Gm_i m_j}{r}$$

where G is Newton's constant, so

$$f_{ij}(r) = \frac{Gm_i m_j}{r^2}.$$

Given inital data $q_i(0), \dot{q}_i(0)$ does there exist a solution of Newton's 2^{nd} law:

$$m\ddot{q}_i(t) = \sum_{j\neq i} -\frac{Gm_i m_j}{|| q_i(t) - q_j(t) ||^2}\frac{q_i(t) - q_j(t)}{|| q_i(t) - q_j(t) ||}$$

for all values of $t \in \mathbb{R}$? Depending on initial data, we may have **collision singularities** where $q_i(t) = q_j(t), (i \neq j)$, and then we can worry if we can continue the solution beyond the singularities. But: no, it is not true that we have a solution for all $q_i(0), \dot{q}_i(0)$. So:

1. Is there a solution for almost everywhere choice of initial data?
   Answer is yes for $n = 2, 3, 4, \ldots$ but open for large n.

2. If there is no collision singularity, then is there a solution for al t?
   Not always! There is a counterexample for $n = 5$:

   $$picture\ of\ 5 - body\ counterexample$$

   Nobody knows of these non-collision singularities arise from a set of initial data of measure zero (if $n \geq 5$).