Classical Mechanics, Lecture 5

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lecture by John Baez

notes by Alex Hoffnung

1 Symmetries and Conserved Quantities — the n-Body Problem

Today we will begin our quest to see where symmetries come from. For this, let us talk about **symmetries and conserved quantities in the n-body problem**.

Consider the problem of n particles in $\mathbb{R}^3$ interacting via central forces. The bodies have positions

$$q_i\colon \mathbb{R} \to \mathbb{R}^3$$

satsifying Newton's 2^{nd} law:

$$F_i(t) \;=\; m_i \ddot{q}_i(t)$$

where $m_i > 0$ and

$$F_i(t) \;=\; \sum_{j \neq i} f_{ij}(||q_i(t) - q_j(t)||) \frac{q_i(t) - q_j(t)}{||q_i(t) - q_j(t)||}$$

where $f_{ij}\colon (0, \infty) \to \mathbb{R}$ satisfying Newton's 3^{rd} law:

$$f_{ij} \;=\; f_{ji}$$

This problem has various symmetries and conserved quantities. The amazing fact about nature is that conserved quantities come from the symmetries! We already know a bunch of conserved quantities. Here is a little chart illustrating it:

Conserved quantities	Symmetries
Energy $E \in \mathbb{R}$	?
Angular momentum $J \in \mathbb{R}^3$	?
Momentum $p \in \mathbb{R}^3$	?

What are some symmetries?

1. Time translation symmetry:
   We can change our mind about when is "t_0" without causing any problems. In other words, if $q_i(t),\ (i = 1, \ldots, n)$ form a solution of $F_i(t) \;=\; m_i \ddot{q}_i(t)$, so do $q_i(t+s)$ for $s \in \mathbb{R}$. Proof: let

   $$\tilde{q}_i(t) \;=\; q_i(t+s)$$

   and show $\tilde{q}_i$ solves $F_i(t) \;=\; m_i \ddot{q}_i(t)$:

$$\begin{aligned}
m_i \tilde{q}_i(t) &= m_i \frac{d^2}{dt^2} q_i(t+s) \\
&= m_i \ddot{q}_i(t+s) \\
&= f_i(t+s) \quad (since\ q_i\ satisfy\ Newton's\ 2^{nd}\ law.) \\
&= \sum_{j \neq i} f_{ij}(||q_i(t+s) - q_j(t+s)||) \frac{q_i(t+s) - q_j(t+s)}{||q_i(t+s) - q_j(t+s)||} \\
&= \sum_{j \neq i} f_{ij}(||\tilde{q}_i(t) - \tilde{q}_j(t)||) \frac{\tilde{q}_i(t) - \tilde{q}_j(t)}{||\tilde{q}_i(t) - \tilde{q}_j(t)||} \\
&= \tilde{F}_i(t)
\end{aligned}$$

2. Time reversal symmetry:
$$\tilde{q}_i(t) \;=\; q_i(-t)$$
Again: q_i satisfying Newton's 2^{nd} law implies $\tilde{q}_i$ also satisfies Newton's 2^{nd} law.

3. Spatial rotation symmetry:
$$\tilde{q}_i(t) \;=\; Rq_i(t)$$
where $R\colon \mathbb{R}^3 \to \mathbb{R}^3$ is a rotation. Again: q_i satisfying Newton's 2^{nd} law implies $\tilde{q}_i$ also does. Then JB breaks the table because he's not strong enough to move the Earth.

4. Spatial translation symmetry:
$$\tilde{q}_i(t) \;=\; q_i(t) \;+\; k, \quad k \in \mathbb{R}^3$$

Again, same thing. 3) and 4) are <u>isometries</u> of $\mathbb{R}^3$, that is, functions $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that
$$||Tx - Ty|| \;=\; ||x - y||, \quad \forall x, y \in \mathbb{R}^3.$$

Theorem 1 *Every isometry $T\colon \mathbb{R}^3 \to]\mathbb{R}^3$ is the composite of:*

1. *a rotation*
2. *a translation*
   and possibly
3. *<u>parity</u> (<u>total spacial inversion</u>):*
$$x \mapsto -x, \quad (x \in \mathbb{R}^3)$$

Let's show that if $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ is an isometry and
$$\tilde{q}_i(t) \;=\; Tq_i(t)$$
then q_i satisfies Newton's 2^{nd} law implies $\tilde{q}_i$ does.

$$\begin{aligned} m\ddot{\tilde{q}}_i(t) &= m\frac{d^2}{dt^2}Tq_i(t) \\ &= m\frac{d}{dt}ddtTq_i(t) \end{aligned}$$

Now use the theorem:
$$Tx \;=\; Sx + k \quad x \in \mathbb{R}^3$$
where $k \in \mathbb{R}^3$ and S is an orthogonal linear transformation (i.e. 3×3 matrix with $SS^\dagger = 1$, or a linear transformation with $||Sx|| = ||x||, \forall x \in \mathbb{R}^3$).
Then
$$\frac{d}{dt}Tq_i(t) \;=\; S\dot{q}_i(t)$$
and
$$\frac{d^2}{dt^2}Tq_i(t) \;=\; S\ddot{q}_i(t)$$

so

$$
\begin{aligned}
m_i\ddot{\tilde{q}}_i(t) &= m_i S\ddot{q}_i(t) \\
&= S\sum_{j\neq i} f(||q_i(t)-q_j(t)||)\frac{q_i(t)-q_j(t)}{||q_i(t)-q_j(t)||} \\
&= \sum_{j\neq i} f(||Tq_i(t)-Tq_j(t)||)\frac{Tq_i(t)-Tq_j(t)}{||Tq_i(t)-Tq_j(t)||}
\end{aligned}
$$

using $Tx = Sx + k$ and that T is an isometry.

There is also a fifth symmetry, **Galilean symmetry**

$$
\tilde{q}_i(t) = q_i(t) + tv, \quad v \in \mathbb{R}^3
$$

If q_i is a solution of Newton's 2^{nd} law, then so is $\tilde{q}_i$.

$$
\begin{aligned}
m\ddot{\tilde{q}}_i(t) &= m\ddot{q}_i(t) \\
&= \sum_{j\neq i} f_i(||q_i(t)-q_j(t)||)\frac{q_i(t)-q_j(t)}{||q_i(t)-q_j(t)||} \\
&= \sum_{j\neq i} f_i(||\tilde{q}_i(t)-\tilde{q}_j(t)||)\frac{\tilde{q}_i(t)-\tilde{q}_j(t)}{||\tilde{q}_i(t)-\tilde{q}_j(t)||}
\end{aligned}
$$

Conserved quantities	Symmetries
Energy $E \in \mathbb{R}$	Time translation symmetry (a 1d group, $\mathbb{R}$)
Angular momentum $J \in \mathbb{R}^3$	Rotation symmetry (a 3d group, SO(3))
Momentum $p \in \mathbb{R}^3$	Translation symmetry (a 3d group, $\mathbb{R}^3$)
?	Galilean symmetry (a 3d group, $\mathbb{R}^3$)