1 The Poisson Bracket

If we have one particle in $\mathbb{R}^n$, we call $\mathbb{R}^n$ the configuration space - the space of possible positions of the whole system. If we have $n$ particles in $\mathbb{R}^3$ the configuration space is $\mathbb{R}^{3n}$, since a point in here is an $n$-tuple $(q_1, \ldots, q_n)$ where $q_i \in \mathbb{R}^3$ is the position of the $i^{th}$ particle.

picture of a pendulum swinging around a pivot

For a pendulum that can swing all the way around in a plane, the configuration space is $S^1$ - the circle. For 42 pendula, the configuration space is

$$S^1 \times \cdots \times S^1 = T^{42}$$

(the 42-dimensional torus). The configuration space of a rigid body in $\mathbb{R}^3$ with center of mass at $0 \in \mathbb{R}^3$ is

$$SO(3) = \{3 \times 3 \text{ rotation matrices}\}$$

$$= \{R: \mathbb{R}^3 \to \mathbb{R}^3 \text{ such that } RR^T = 1, \det R = 1\}$$

a 3-dim manifold. The configuration space of a rigid body in $\mathbb{R}^3$ is $SO(3) \times \mathbb{R}^3$, where a point in $\mathbb{R}^3$ specifies the center of mass.

But Hamiltonian mechanics focuses not on the configuration space but on the phase space or state space, where a point specifies the position and momentum of the system. For a single particle in $\mathbb{R}^n$, the phase space is:

$$\mathbb{R}^{2n} \cong \mathbb{R}^n \times \mathbb{R}^n \in (q, p)$$

where $q \in \mathbb{R}^n$ is the position and $p \in \mathbb{R}^n$ is the momentum. For $n$ particles in $\mathbb{R}^3$, the phase space is

$$\mathbb{R}^{6n} \cong \mathbb{R}^{3n} \times \mathbb{R}^{3n}.$$

For a rigid body in $\mathbb{R}^3$ with center of mass at the origin, the phase space is $SO(3) \times \mathbb{R}^3$ where $q \in SO(3)$ is the position and $p \in \mathbb{R}^3$ is actually the angular momentum, usually called $J$. Actually a better description is $T^*SO(3)$ - the cotangent bundle of SO(3).

In our example of a particle in $\mathbb{R}^n$, we said an observable is a smooth function

$$F: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$$

by

$$(q, p) \mapsto F(q, p)$$

from the phase space to $\mathbb{R}$, sending each point of phase space (or state of our system) to the value of the observables.

If $X$ is any manifold, thought of as a phase space, let

$$C^\infty(X) = \{\text{smooth functions } F: X \to \mathbb{R}\}$$

be the set of observables. This is a commutative algebra.
Definition 1 A commutative algebra is a (real) vector space $A$ equipped with a product satisfying:

1. bilinearity:

\[(\alpha F + \beta G)H = \alpha FH + \beta GH, \quad \forall \alpha, \beta \in \mathbb{R}\]
\[F(\alpha G + \beta H) = \alpha FG + \beta FH, \quad F, G, H \in A\]

2. commutativity:

\[FG = GF\]

3. associativity:

\[(FG)H = F(GH)\]

$C^\infty(X)$ becomes a commutative algebra with the “obvious” addition, scalar multiplication, and multiplication:

\[(\alpha F)(x) = \alpha F(x), \quad x \in X\]
\[(F + G)(x) = F(x) + G(x)\]
\[(FG)(x) = F(x)G(x)\]

But, in our example $X = \mathbb{R}^{2n}$, there’s another operation, the Poisson bracket:

\[
\{F, G\} = \sum_{i=1}^{n} \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} - \frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i}
\]

This makes $C^\infty(X)$ into a ‘Lie algebra’:

**Definition 2** A Lie algebra is a (real) vector space $A$ equipped with a “Lie bracket” $\{\cdot, \cdot\}$

1. bilinearity:

\[\{\alpha F + \beta G, H\} = \alpha\{F, H\} + \beta\{G, H\}\]
\[\{F, \alpha G + \beta H\} = \alpha\{F, G\} + \beta\{G, H\}\]

2. antisymmetry:

\[\{F, G\} = -\{G, F\}\]

3. Jacobi identity:

\[\{F, \{G, H\}\} = \{\{F, G\}, H\} + \{G, \{F, H\}\}\]

(Rule 3 looks like the product rule: $dGH = (dG)H + g(dH)$.)

In fact for all classical mechanics problems, the algebra of observables $C^\infty(X)$ is always both a commutative algebra and a Lie algebra, but even better, they fit together to form a Poisson algebra.

**Definition 3** A Poisson algebra is a vector space $A$ with a product making it into a commutative algebra and a bracket $\{\cdot, \cdot\}$ making it into a Lie algebra such that

\[\{F, GH\} = \{F, G\}H + G\{F, H\}\]
We saw that for a particle in \( \mathbb{R}^n \), with energy given by:

\[
H(q,p) = \frac{p^2}{2m} + V(q)
\]

where \( V: \mathbb{R}^n \to \mathbb{R} \), Newton’s 2\(^{nd}\) law can be rewritten as Hamilton’s equations:

\[
\frac{dq_i}{dt}(t) = \frac{\partial H}{\partial p_i}(q(t), p(t))
\]

\[
\frac{dp_i}{dt}(t) = -\frac{\partial H}{\partial q_i}(q(t), p(t))
\]

If \( H \in C^\infty(\mathbb{R}^{2n}) \) is nice, these have a unique smooth solution for any choice of initial \( q(0) = q \) and \( p(0) = p \). Then we get a function

\[
\phi: \mathbb{R} \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}
\]

by

\[(t, q, p) \mapsto (q(t), p(t))\]

which describes time evolution. Often we write

\[
\phi(t, q, p) = \phi_t(q, p)
\]

where

\[
\phi_t: \mathbb{R}^{2n} \to \mathbb{R}^{2n}.
\]

Then we can say how any observable changes with time: given \( F \in C^\infty(\mathbb{R}^{2n}) \) we get

\[
F_t(q, p) = F(\phi_t(q, p))
\]

and Hamilton’s equations say:

\[
\frac{d}{dt} F_t = \sum_{i=1}^n \frac{\partial F}{\partial q_i} \frac{dq_i}{dt} + \frac{\partial F}{\partial p_i} \frac{dp_i}{dt}
\]

\[
= \sum_{i=1}^n \frac{\partial H}{\partial p_i} \frac{\partial F}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial F}{\partial p_i}
\]

\[
= \{H, F\}_t
\]

This is why the Poisson algebra is important.