1 Poisson Manifolds

We have described Poisson brackets for functions on $\mathbb{R}^{2n}$ - the phase space for a system whose configuration space is $\mathbb{R}^n$. Now let’s generalize this to systems whose configuration space is any manifold, $M$. Here we will see that the phase space is the “cotangent bundle” $T^*M$ and this is a “Poisson manifold” - a manifold such that the commutative algebra of smooth real-valued functions on it, $C^\infty(T^*M)$ is equipped with Poisson bracket $\{\cdot,\cdot\}$ making it into a Poisson algebra - the Poisson algebra of “observables” for our system.

Example: a particle on a sphere $S^2$.

The position and momentum of this particle give a point in $T^*S^2 : q \in S^2, p \in T_q^*M$, so $(q, p) \in T^*S^2$. Recall that a manifold $M$ is a topological space such that every point $q \in M$ has a “neighborhood that looks like $\mathbb{R}^n$.” In other words, there is an open set $U \subset M$ with $q \in U$ and a bijection $\phi: U \to \mathbb{R}^n$.

Indeed we have a collection of these $(U_i, \phi_i: U_i \to \mathbb{R}^n)$ and they are compatible:

that is, $\phi_j \circ \phi_i^{-1}$ is smooth (infinitely differentiable) where defined. A collection of this sort is an atlas, and the functions $\phi_i: U_i \to \mathbb{R}^n$ are called charts. We will usually use a maximal atlas, i.e. one containing all charts that are compatible with all charts in the atlas. So - a manifold is a topological space with a maximal atlas.

If the manifold $M$ is the configuration space of some physical system, the a point $q \in M$ describes the position of the system and a “tangent vector” $v \in T_q^*M$ describes its velocity, where $T_q^*M$ is the tangent space of $M$ at $q$:

which can be defined in various ways:

1. A tangent vector $v$ at the point $q$ is an equivalence class of (smooth) curves $\gamma: \mathbb{R} \to M$ such that $\gamma(0) = q$, where $\gamma_1 \sim \gamma_2$ if and only if for every smooth function $f \in C^\infty(M)$ (smooth real-valued functions on $M$) we have

$$\frac{d}{dt}f(\gamma_1(t))|_{t=0} = \frac{d}{dt}f(\gamma_2(t))|_{t=0}.$$

You can show the set of such equivalence classes is an $n$-dimensional vector space, the tangent space $T_q^*M$. Given $v = [\gamma] \in T_q^*M$, we can define the derivative $vf \in \mathbb{R}$ for any $f \in C^\infty(M)$ by

$$vf = \frac{d}{dt}f(\gamma(t))|_{t=0}.$$

2. A tangent vector $v$ at $q \in M$ is a derivation $v: C^\infty(M) \to \mathbb{R}$

i.e., a map that is:
\begin{itemize}
  \item \(v(\alpha f + \beta g) = \alpha v(f) + \beta v(g)\) - linearity
  \item \(v(fg) = v(f)g(q) + f(q)v(g)\) - product rule
\end{itemize}

These clearly form a vector space, the **tangent space** \(T_q M\).

The set of all position-velocity pairs is a manifold, the **tangent bundle** of \(M\):

\[ TM = \{(q, v) : q \in M, v \in T_q M\} \]

Naively, we might define momentum by \(p = mv\), in which case it would be a tangent vector. It is better to think of it as a “cotangent vector”. Every vector space \(V\) has a **dual** \(V^*\):

\[ V^* = \{l: V \rightarrow \mathbb{R} : l \text{ linear}\} \]

The **cotangent space** of \(M\) at \(q \in M\) is:

\[ T_q^* M = (T_q M)^* \]

and the **cotangent bundle** of \(M\) is:

\[ T^* M = \{(q, p) : q \in M, p \in T_q^* M\} \]

So \(T^* M\) will be the system’s “phase space” - space of position-momentum pairs.

What good are cotangent vectors, though?

The “gradient” or “differential” of a function \(f \in C^\infty(M)\) at \(q \in M\) is a cotangent vector, \((df)_q \in T_q^* M\):

\[ (df)_q (v) = v(f), v \in T_q M \]

(or in low-brow notation: \((\nabla f)(q) \cdot v = vf\).

The potential energy for our system (e.g. a particle on \(M\)) is some function of its position: \(V \in C^\infty(M)\). We have seen already that “\(\nabla V = -F\)” - but this really means

\[ (dV)_q = -F(q) \]

where \(F(q)\), the **force** at \(q \in M\), is a cotangent vector: \(F(q) \in T_q^* M\).

A tangent vector looks like a little arrow:

(picture of a tangent vector)

A cotangent vector looks like a “stack of hyperplanes” - its level surfaces:

(picture of level surfaces)

Together they give a number \(l(v) \in \mathbb{R}\):

(picture of tangent vector crossing level curves)

In physics, the velocity \(v \in T_q M\) is a tangent vector and the force \(F \in T_q^* M\) is a cotangent vector, so \(F(v) \in \mathbb{R}\). We have seen this before in the formula:

\[ \int_{t_1}^{t_2} F \cdot \dot{q}(t) dt = \text{work} \]

Now we would say, given a particle’s path \(q: \mathbb{R} \rightarrow M\), that work from \(t_1\) to \(t_2\) is equal to

\[ \int_{t_1}^{t_2} F(q(t))(\dot{q}(t))dt \]

Since force is a cotangent vector, so is momentum, since:

\[ \frac{dp}{dt} = F \]

So: a position-momentum pair \((q, p)\) is really a point in \(T^* M\):

\[ q \in M, p \in T_q^* M. \]