Classical Mechanics, Lecture 8

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1 Poisson Manifolds

We have described Poisson brackets for functions on $\mathbb{R}^{2n}$ - the phase space for a system whose configuration space is $\mathbb{R}^n$. Now let's generalize this to systems whose configuration space is any manifold, M. Here we will see that the phase space is the "cotangent bundle" T^*M and this is a "Poisson manifold" - a manifold such that the commutative algebra of smooth real-valued functions on it, $C^\infty(T^*M)$ is equipped with Poisson bracket $\{\cdot,\cdot\}$ making it into a Poisson algebra - the Poisson algebra of "observables" for our system.

Example: a particle on a sphere S^2.
(picture of a sphere $M = S^2$ with point $q \in M$)
The position and momentum of this particle give a point in $T^*S^2 : q \in S^2, p \in T_q^*M$, so $(q,p) \in T^*S^2$.
Recall that a manifold M is a topological space such that every point $q \in M$ has a "neighborhood that looks like $\mathbb{R}^n$." In other words, there is an open set $U \subset M$ with $q \in U$ and a bijection

$$\phi\colon U \to \mathbb{R}^n.$$

Indeed we have a collection of these $(U_i, \phi_i\colon U_i \to \mathbb{R}^n)$ and they are **compatible**:
(picture of charts overlapping)
that is, $\phi_j \circ \phi_i^{-1}$ is smooth (infinitely differentiable) where defined. A collection of this sort is an **atlas**, and the functions $\phi_i\colon U_i \to \mathbb{R}^n$ are called **charts**. We will usually use a **maximal** atlas, i.e. one containing all charts that are compatible with all charts in the atlas. So - a **manifold** is a topological space with a maximal atlas.

If the manifold M is the configuration space of some physical system, the a point $q \in M$ describes the position of the system and a "tangent vector" $v \in T_qM$ describes its velocity, where T_qM is the **tangent space** of M at q:
(picture of tangent space to S^2 at q)
which can be defined in various ways:

1. A **tangent vector** v at the point q is an equivalence class of (smooth) curves

   $$\gamma\colon \mathbb{R} \to M$$

   such that $\gamma(0) = q$, where $\gamma_1 \sim \gamma_2$ if and only if for every smooth function $f \in C^\infty(M)$ (smooth real-valued functions on M) we have

   $$\frac{d}{dt} f(\gamma_1(t))|_{t=0} = \frac{d}{dt} f(\gamma_2(t))|_{t=0}.$$

   You can show the set of such equivalence classes is an n-dimensional vector space, the **tangent space** T_qM. Given $v = [\gamma] \in T_qM$, we can define the **derivative** $vf \in \mathbb{R}$ for any $f \in C^\infty(M)$ by

   $$vf = \frac{d}{dt} f(\gamma(t))|_{t=0}$$

2. A tangent vector v at $q \in M$ is a **derivation**

   $$v\colon C^\infty(M) \to \mathbb{R}$$

   i.e., a map that is:

- $v(\alpha f + \beta g) = \alpha v(f) + \beta v(g)$ - linearity
- $v(fg) = v(f)g(q) + f(q)v(g)$ - product rule

These clearly form a vector space, the **tangent space** T_qM.

The set of all position-velocity pairs is a manifold, the **tangent bundle** of M:

$$TM = \{(q, v) : q \in M, v \in T_qM\}$$

Naively, we might define momentum by $p = mv$, in which case it would be a tangent vector. It is better to think of it is a "cotangent vector". Every vector space V has a **dual** V^*:

$$V^* = \{l\colon V \to \mathbb{R} : l \ linear\}$$

The **cotangent space** of M at $q \in M$ is:

$$T_q^*M = (T_qM)^*$$

and the **cotangent bundle** of M is:

$$T^*M = \{(q, p) : q \in M, p \in T_q^*M\}$$

So T^*M will be the system's "phase space" - space of position-momentum pairs.

What good are cotangent vectors, though?
The "gradient" or "differential" of a function $f \in C^\infty(M)$ at $q \in M$ is a cotangent vector, $(df)_q \in T_q^*M$:

$$(df)_q(v) = v(f), v \in T_qM$$

(or in low-brow notation: $(\nabla f)(q) \cdot v = vf$).
The potential energy for our system (e.g. a particle on M) is some fucntion of its position: $V \in C^\infty(M)$. We have seen already that "$\nabla V = -F$" - but this really means

$$(dV)_q = -F(q)$$

where $F(q)$, the **force** at $q \in M$, is a cotangent vector: $F(q) \in T_q^*M$.
A tangent vector looks like a little arrow:
(picture of a tangent vector)
A cotangent vector looks like a "stack of hyperplanes" - its level surfaces:
(picture of level surfaces)
Together they give a number $l(v) \in \mathbb{R}$:
(picture of tangent vector crossing level curves)

In physics, the velocity $v \in T_pM$ is a tangent vector and the force $F \in T_q^*M$ is a cotangent vector, so $F(v) \in \mathbb{R}$. We have seen this before in the formula:

$$\int_{t_1}^{t_2} F \cdot \dot{q}(t)dt = work$$

Now we would say, given a particle's path $q\colon \mathbb{R} \to M$, that work from t_1 to t_2 is equal to

$$\int_{t_1}^{t_2} F(q(t))(\dot{q}(t))dt$$

Since force is a cotangent vector, so is momentum, since:

$$\frac{dp}{dt} = F$$

So: a position-momentum pair (q, p) is really a point in T^*M:

$$q \in M, p \in T_q^*M.$$