Classical Mechanics, Lecture 9

February 7, 2008

lecture by John Baez

notes by Alex Hoffnung

1 Poisson Manifolds

Let M be any n-dimensional manifold - the configuration space of some classical system, for example a particle on M. Then the phase space is the cotangent bundle of M:

$$T^*M = \{q \in M, p \in T_q^*M\}$$

Let's see how this is a Poisson manifold:

Definition 1 *A* **Poisson manifold** X *is a manifold with a bracket operation*

$$\{\cdot,\cdot\}\colon C^\infty(X) \times C^\infty(X) \to C^\infty(X)$$

making the commutative algebra

$$C^\infty(X) = \{f\colon X \to \mathbb{R} : f \text{ smooth}\}$$

into a Poisson algebra.

Example: $M = \mathbb{R}^n$

In this case $\mathbb{R}^n$ has coordinates $x_i\colon \mathbb{R}^n \to \mathbb{R}$, so for each point in $q \in \mathbb{R}^n$ we get a basis of $T_q\mathbb{R}^n$, namely:

$$\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}.$$

(Picture of $\mathbb{R}^2$ with coordinates x_1, x_2, and a tangent plane at q with basis.)

These are tangent vectors: given $f \in C^\infty(\mathbb{R}^n)$, they act on it to give a number:

$$\frac{\partial f}{\partial x_i}(q) \in \mathbb{R}$$

We also get a basis of $T_q^*\mathbb{R}^n$, namely:

$$dx_1, \ldots, dx_n.$$

(Picture of $\mathbb{R}^2$ with coordinates x_1, x_2, and a cotangent space at q with basis.)

(Recall, given $f \in C^\infty(\mathbb{R}^n)$, we get $(df)_q \in T)q^*\mathbb{R}^n$ by:

$$(df)_q(v) = v(f)(q), \forall f \in C^\infty(\mathbb{R}^n)$$

We can call this just "*df*" if we are feeling lazy.)

Note:

$$\begin{aligned} (dx_i)(\frac{\partial}{\partial x_j}) &= \frac{\partial}{\partial x_j} x_i \\ &= \delta_{ij} \end{aligned}$$

so dx_i is the "dual basis" to $\frac{\partial}{\partial x_i}$.

Using this standard basis for $T_q^*\mathbb{R}^n$ we get an isomorphism

$$T_q^*\mathbb{R}^n \cong \mathbb{R}^n$$

$$dx_i \mapsto (0, \ldots, 1, \ldots, 0)$$

with 1 in the n^{th} slot. So we get an isomorphism

$$\begin{aligned} T^*\mathbb{R}^n &= \{q \in \mathbb{R}^n, p \in T_q^*\mathbb{R}^n\} \\ &\cong \{q \in \mathbb{R}^n, p \in \mathbb{R}^n\} \\ &\cong \mathbb{R}^n \times \mathbb{R}^n \end{aligned}$$

This lets is put coordinates on $T^*\mathbb{R}^n$, namey

$$q_i, p_i \colon T^*\mathbb{R}^n \to \mathbb{R}, i = 1, \ldots, n$$

This lets us make $T^*\mathbb{R}^n$ into a Poisson manifold:

$$\{F, G\} = \sum_{i=1}^{n} \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i} - \frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i}$$

(using your homework).
More generally, suppose M is any n-dimensional manifold. Given any $q \in M$ we can find an open set $U \ni x$ and a chart:

$$\phi \colon U \to \mathbb{R}^n$$

This gives coordinates $x_i \circ \phi$ on $\mathbb{R}^n$, which we just call x_i for short. Copying what we did, we get coordinates q_i, p_i on

$$T^*U = \{q \in U, p \in T_q^*U\}$$

and if $q \in U$, then $T_q^*U = T_q^*M$. How do we make T^*M into a Poisson manifold? Given $F, G \in C^\infty(T^*M)$, we define $\{F, G\}$ on $T^*U \subseteq T^*M$ by:

$$\{F, G\} = \sum_{i=1}^{n} \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i} - \frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i}.$$

Now, alas, we need to check that the Poisson brackets are well-defined on all of T^*M - i.e., independent of the choice of chart. But, let's not. It will be easier to define the Poisson brackets in a coordinate-free way later. First we will develop some more geometry and start understanding what Poisson brackets mean.

2 More Differential Geometry

Given manifolds M and N, a function $: M \to N$ is called **smooth**, or a **map**, if any of these hold:

1. Given any charts $\phi \colon U \to \mathbb{R}^m$ with $U \subseteq M$, $\psi \colon V \to \mathbb{R}^n$ with $V \subseteq N$, this composite

$$\mathbb{R}^m \to U \subseteq M \to N \supseteq V \to \mathbb{R}^n$$

   is smooth where defined. It's enough to check this for one chart U containing each point $q \in M$ and one chart containing each point $f(q) \in N$.

2. Given any smooth curve $\gamma \colon \mathbb{R} \to M$, $f \circ \gamma \colon \mathbb{R} \to N$ is a smooth curve in N.

3. Given any $g \in C^\infty(N)$, then $g \circ f \in C^\infty(M)$.

We can define a vector field on M in two equivalent ways:

1. A smooth map $V\colon M \to TM$ such that $v(q) \in T_q M$.
2. A **derivation** $D\colon C^\infty(M) \to C^\infty(M)$, i.e., a linear map:

   $$D(\alpha f + \beta g) = \alpha Df + \beta Dg, \alpha, \beta \in \mathbb{R}$$

   satisfying the product rule (or **Leibniz** law):

   $$D(fg) = D(f)g + fDg.$$

Given a derivation $D\colon C^\infty(M) \to C^\infty(M)$ we get $v\colon M \to TM$ by:

$$(v(q)f) = (Df)(q), q \in M, f \in C^\infty(M)$$

and conversely. This is relevant to Poisson manifolds, since it means

$$\{F, -\}\colon C^\infty(X) \to C^\infty(X)$$

is a vector field for any Poisson manifold X and $F \in C^\infty(X)$. So in classical mechanics, observables give vector fields on phase space!

(picture of X with Hamiltonian level curves for harmonic oscillator with a vector field given by Poisson bracket and energy - time evolution! The vectors are tangent to the level curves due to conservation of energy.)
For example, the observable “energy” gives a vector field describing time evolution: as time passes, the state of the system $\gamma(t) \in X$ moves in the direction of this vector field! Even better, it moves along level curves of the energy function!