1 Poisson Manifolds

Let $M$ be any $n$-dimensional manifold - the configuration space of some classical system, for example a particle on $M$. Then the phase space is the cotangent bundle of $M$:

$$T^*M = \{ q \in M, p \in T^*_q M \}$$

Let’s see how this is a Poisson manifold:

**Definition 1** A Poisson manifold $X$ is a manifold with a bracket operation

$$\{ \cdot, \cdot \} : C^\infty(X) \times C^\infty(X) \to C^\infty(X)$$

making the commutative algebra

$$C^\infty(X) = \{ f : X \to \mathbb{R} : f \text{ smooth} \}$$

into a Poisson algebra.

**Example**: $M = \mathbb{R}^n$

In this case $\mathbb{R}^n$ has coordinates $x_i : \mathbb{R}^n \to \mathbb{R}$, so for each point in $q \in \mathbb{R}^n$ we get a basis of $T_q^* \mathbb{R}^n$, namely:

$$\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}.$$  

(Picture of $\mathbb{R}^2$ with coordinates $x_1, x_2$, and a tangent plane at $q$ with basis.)

These are tangent vectors: given $f \in C^\infty(\mathbb{R}^n)$, they act on it to give a number:

$$\frac{\partial f}{\partial x_i}(q) \in \mathbb{R}$$

We also get a basis of $T_q^* \mathbb{R}^n$, namely:

$$dx_1, \ldots, dx_n.$$  

(Picture of $\mathbb{R}^2$ with coordinates $x_1, x_2$, and a cotangent space at $q$ with basis.)

(Recall, given $f \in C^\infty(\mathbb{R}^n)$, we get $(df)_q \in T_q^* \mathbb{R}^n$ by:

$$(df)_q(v) = v(f)(q), \forall f \in C^\infty(\mathbb{R}^n)$$

We can call this just “$df$” if we are feeling lazy.)

**Note**: 

$$(dx_i)\left( \frac{\partial}{\partial x_j} \right) = \frac{\partial}{\partial x_j} x_i = \delta_{ij}$$

so $dx_i$ is the “dual basis” to $\frac{\partial}{\partial x_i}$.

Using this standard basis for $T_q^* \mathbb{R}^n$ we get an isomorphism

$$T_q^* \mathbb{R}^n \cong \mathbb{R}^n$$
with 1 in the $n^{th}$ slot. So we get an isomorphism
\[ T^*\mathbb{R}^n = \{ q \in \mathbb{R}^n, p \in T_q^*\mathbb{R}^n \} \]
\[ \cong \{ q \in \mathbb{R}^n, p \in \mathbb{R}^n \} \]
\[ \cong \mathbb{R}^n \times \mathbb{R}^n \]
This lets us put coordinates on $T^*\mathbb{R}^n$, namely
\[ q_i, p_i: T^*\mathbb{R}^n \to \mathbb{R}, i = 1, \ldots, n \]
This lets us make $T^*\mathbb{R}^n$ into a Poisson manifold:
\[ \{F, G\} = \sum_{i=1}^{n} \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} - \frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} \]
(\text{using your homework}).
More generally, suppose $M$ is any $n$-dimensional manifold. Given any $q \in M$ we can find an open set $U \ni x$ and a chart:
\[ \phi: U \to \mathbb{R}^n \]
This gives coordinates $x_i \circ \phi$ on $\mathbb{R}^n$, which we just call $x_i$ for short. Copying what we did, we get coordinates $q_i, p_i$ on
\[ T^*U = \{ q \in U, p \in T_q^*U \} \]
and if $q \in U$, then $T_q^*U = T_q^*M$. How do we make $T^*M$ into a Poisson manifold? Given $F, G \in C^\infty(T^*M)$, we define $\{F, G\}$ on $T^*U \subseteq T^*M$ by:
\[ \{F, G\} = \sum_{i=1}^{n} \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} - \frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} \]
Now, alas, we need to check that the Poisson brackets are well-defined on all of $T^*M$ - i.e., independent of the choice of chart. But, let’s not. It will be easier to define the Poisson brackets in a coordinate-free way later. First we will develop some more geometry and start understanding what Poisson brackets mean.

2 More Differential Geometry

Given manifolds $M$ and $N$, a function $: M \to N$ is called smooth, or a map, if any of these hold:

1. Given any charts $\phi: U \to \mathbb{R}^n$ with $U \subseteq M$, $\psi: V \to \mathbb{R}^n$ with $V \subseteq N$, this composite
\[ \mathbb{R}^n \to U \subseteq M \to N \supseteq V \to \mathbb{R}^n \]
is smooth where defined. It’s enough to check this for one chart $U$ containing each point $q \in M$ and one chart containing each point $f(q) \in N$.

2. Given any smooth curve $\gamma: \mathbb{R} \to M, f \circ \gamma: \mathbb{R} \to N$ is a smooth curve in $N$.

3. Given any $g \in C^\infty(N)$, then $g \circ f \in C^\infty(M)$.
We can define a vector field on $M$ in two equivalent ways:
1. A smooth map $V: M \to TM$ such that $v(q) \in T_qM$.

2. A derivation $D: C^\infty(M) \to C^\infty(M)$, i.e., a linear map:

$$D(\alpha f + \beta g) = \alpha Df + \beta Dg, \alpha, \beta \in \mathbb{R}$$

satisfying the product rule (or Leibniz law):

$$D(fg) = D(f)g + fDg.$$

Given a derivation $D: C^\infty(M) \to C^\infty(M)$ we get $v: M \to TM$ by:

$$(v(q)f) = (Df)(q), q \in M, f \in C^\infty(M)$$

and conversely. This is relevant to Poisson manifolds, since it means

$$\{F, -\}: C^\infty(X) \to C^\infty(X)$$

is a vector field for any Poisson manifold $X$ and $F \in C^\infty(X)$. So in classical mechanics, observables give vector fields on phase space!

(picture of $X$ with Hamiltonian level curves for harmonic oscillator with a vector field given by Poisson bracket and energy - time evolution! The vectors are tangent to the level curves due to conservation of energy.)

For example, the observable “energy” gives a vector field describing time evolution: as time passes, the state of the system $\gamma(t) \in X$ moves in the direction of this vector field! Even better, it moves along level curves of the energy function!)