Classical Mechanics

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28 March 2005

The Lagrangian Approach to Classical Mechanics

Let's compare the Newtonian approach:

$$F = ma$$

to the much subtler approach of Lagrangian mechanics.
In the Newtonian approach, let's consider a particle moving in $\mathbb{R}^n$ ("space"). Its position depends on time $t \in \mathbb{R}$, so it defines a function

$$q : \mathbb{R} \longrightarrow \mathbb{R}^n$$

From this we can define <u>velocity</u>

$$v = \dot{q} : \mathbb{R} \to \mathbb{R}^n$$

where $\dot{q} = \frac{dq}{dt}$, and also <u>acceleration</u>:

$$a = \ddot{q} : \mathbb{R} \to \mathbb{R}^n$$

Let $m > 0$ be the <u>mass</u> of the particle, & let F be the vector field on $\mathbb{R}^n$, called the <u>force</u>.

Newton claimed that q would satisfy $F = ma$, i.e.

$$m\,a(t) = F(q(t))$$

This is a 2nd-order ODE for $q : \mathbb{R} \to \mathbb{R}^n$ which will have a unique solution given $q(t_0)$ & $\dot{q}(t_0)$, at least if the vector field F is nice enough (e.g. smooth & bounded: $\|F(x)\| < M$ for some $M > 0$, all $x \in \mathbb{R}^n$)

It turns out that the kinetic energy

$$K(t) = \frac{1}{2} m v(t) \cdot v(t)$$

is very interesting, because

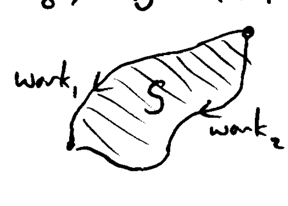
$$\begin{aligned} \frac{d}{dt} K(t) &= m\, v(t) \cdot a(t) \\ &= F(q(t)) \cdot v(t) \end{aligned}$$

so kinetic energy changes depending on whether you push towards or against the direction $v(t)$. Moreover,

$$\begin{aligned} K(t_1) - K(t_0) &= \int_{t_0}^{t_1} F(q(t)) \cdot v(t)\, dt \\ &= \int_{t_0}^{t_1} F(q(t)) \cdot \dot{q}(t)\, dt \end{aligned}$$

so the change of kinetic energy is equal to the work done by the force, i.e. the integral of F along the curve $q : [t_0, t_1] \to \mathbb{R}^n$. This implies that the change in kinetic energy $K(t_1) - K(t_0)$ is independent of the curve going from $q(t_0) = a$ to $q(t_0) = b$ iff

$$\nabla \times F = 0$$



$\text{work}_1 - \text{work}_2 = \int_S \nabla \times F$

which in turn is true iff

$$F = -\nabla V$$

for some function $V : \mathbb{R}^n \to \mathbb{R}$ which is unique up to an additive constant. Such forces are called <u>conservative</u> because in these cases we can define the <u>energy</u> of the particle

$$E(t) = K(t) + V(q(t))$$

where $V(t) := V(q(t))$ is called the <u>potential energy</u> & we can see that E is <u>conserved</u>, i.e. constant as a function of time, given $F=ma$:

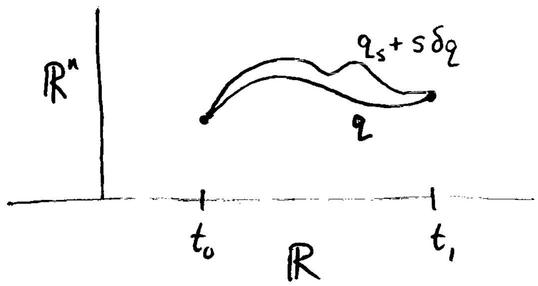
$$\frac{d}{dt}\Big[K(t) + V(q(t))\Big] = F(q(t))\cdot v(t) + \nabla V(q(t))\cdot v(t)$$
$$= 0 \quad \text{since} \quad F = -\nabla V.$$

Conservative forces allow a whole battery of new techniques, especially Hamiltonian and Lagrangian methods. In the Lagrangian approach we define a quantity

$$L = K(t) - V(q(t))$$

called the <u>Lagrangian</u> and for any curve $q : [t_0, t_1] \to \mathbb{R}^n$ with $q(t_0) = a$, $q(t_1) = b$ we define the <u>action</u>

$$S(q) = \int_{t_0}^{t_1} L(t)\, dt$$



We can find which curve(s) like this satisfies $F = ma$ by finding curves that are critical points of S, i.e. curves such that

$$\frac{d}{ds} S(q_s)\Big|_{s=0} = 0$$

where

$$q_s = q + s\,\delta q$$

for all $\delta q : [t_0, t_1] \to \mathbb{R}^n$ with $\delta q(t_0) = \delta q(t_1) = 0$.

Let's show

$$F = ma \iff \frac{d}{ds} S(q_s)\Big|_{t=0} = 0 \quad \forall\, \delta q : [t_0, t_1] \to \mathbb{R}^n \quad \delta q(t_0) = \delta q(t_1) = 0.$$

$$(v_s := \dot{q}_s)$$

$$\frac{d}{ds} S(q_s)\Big|_{s=0} = \frac{d}{ds} \int_{t_0}^{t_1} \frac{1}{2} m\, v_s(t) \cdot v_s(t) - V(q_s(t))\, dt \Big|_{s=0}$$

smoothness of functions

$$= \int_{t_0}^{t_1} \frac{d}{ds}\left(\frac{1}{2} m v_s(t) - V(q_s(t))\right) dt \Big|_{s=0}$$

$$= \int_{t_0}^{t_1} m\, v_s(t) \cdot \frac{d}{ds} v_s(t) - \nabla V(q_s(t)) \cdot \frac{d}{ds} q_s(t)\, dt \Big|_{s=0}$$

$$= \int_{t_0}^{t_1} m\dot{q}_s(t) \frac{d}{dt}\frac{d}{ds} q_s(t) - \nabla V(q_s(t)) \cdot \underbrace{\frac{d}{ds} q_s(t)}_{\delta q(t)} \, dt \Big|_{s=0}$$

& ~~by~~ integration by parts

$$= \int_{t_0}^{t_1} \left(-m\ddot{q}_s(t) - \nabla V(q_s(t))\right) \frac{d}{ds} q_s(t) \, dt \Big|_{s=0}$$

noting that boundary terms vanish since $\delta q = 0$ at t_0, t_1. This is zero for all δq iff

$$-m\ddot{q}(t) - \nabla V(q(t)) = 0$$

i.e.

$$F = ma.$$

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30 March 2005

The Prehistory of the Lagrangian Approach

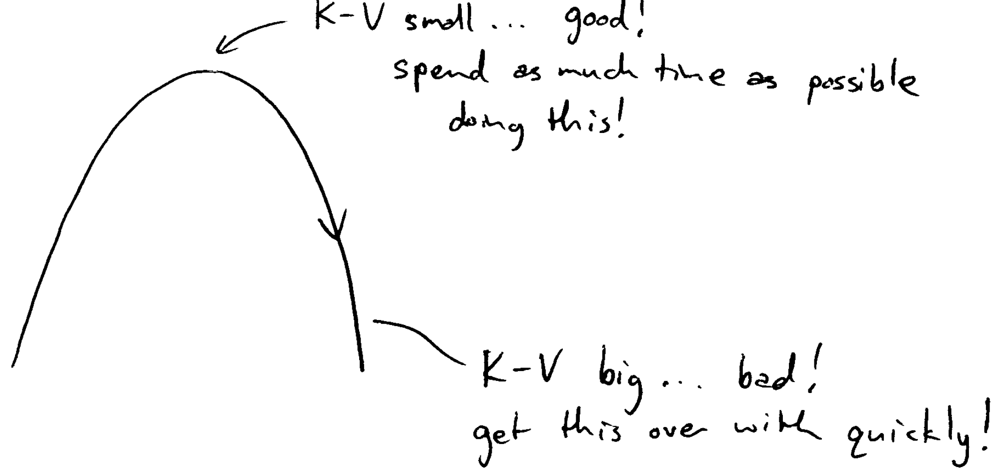
We've seen that a particle going from point a at time t_0 to a point b at time t_1 follows the path(s) that are critical points of the action:

$$S = \int_{t_0}^{t_1} K - V \, dt$$

so, slight changes in its path don't change the action (to first order).

Often, but not always, the action is minimized, so this is called the Principle of Least Action. Why does nature like to minimize action? And why *this* action: $\int K-V\,dt$? Note $K+V$ is conserved, so energy sloshes back and forth between "actual" (kinetic) & potential forms. So the Lagrangian $L = K-V$ measures "happeningness − potential to happen", & we're seeing nature likes to minimize the total of this over time.

For example: Projectile motion

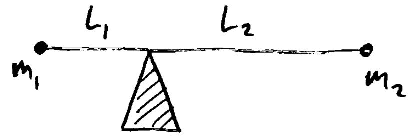


Another related question: how did people discover this principle? Let's look at two predecessors:

1. the "Principle of Least Energy"

Before dynamics, physicists studied statics: the study of objects at rest, or "in equilibrium."

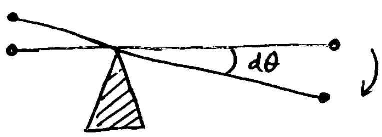
Archimedes studied a see-saw, or lever



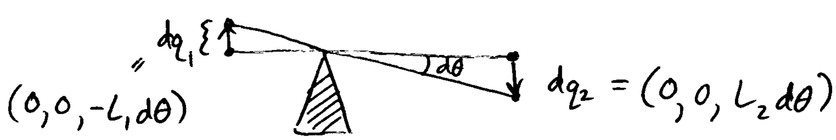
He saw that this would be in equilibrium if

$$m_1 L_1 = m_2 L_2$$

"Give me a place to stand & I will move the world."
Later D'Alembert understood this using his "Principle of Virtual Work." He considered moving the lever slightly – i.e. infinitesimally



He claimed that in equilibrium the (infinitesimal) work done by this motion is zero.



& the work done on the ith body is

$$dW_i = F_i dq_i$$

& gravity pulls down with force $m_i g$ (g = gravitational const)

so

$$dW_1 = (0,0,-mg_1)\cdot(0,0,-L_1\,d\theta)$$
$$= m_1 g L_1\,d\theta$$

& similarly

$$dW_2 = -m_2 g L_2\,d\theta$$

D'Alembert's principle says that equilibrium occurs when the "virtual work" $dW = dW_1 + dW_2$ vanishes for all $d\theta$ (all possible infinitesimal motions). This happens when

$$m_1 L_1 - m_2 L_2 = 0$$

just as Archimedes said.

(Note: here we have 2 particles in $\mathbb{R}^3$ subject to a constraint. Certainly n particles in $\mathbb{R}^3$ can be treated as a single "particle" in $\mathbb{R}^{3n}$, and if there are constraints it can move on some submanifold of $\mathbb{R}^{3n}$. So ultimately we need to study a particle on an arbitrary <u>manifold</u>)

For a particle in $\mathbb{R}^n$, D'Alembert's principle simply says:

$q(t) = q_0$ satisfies $F = ma$ (the particle is in equilibrium)

$\Updownarrow$

$dW = F\cdot dq$ vanishes for all $dq \in \mathbb{R}^n$ (virtual work vanishes for all infinitesimal displacements)

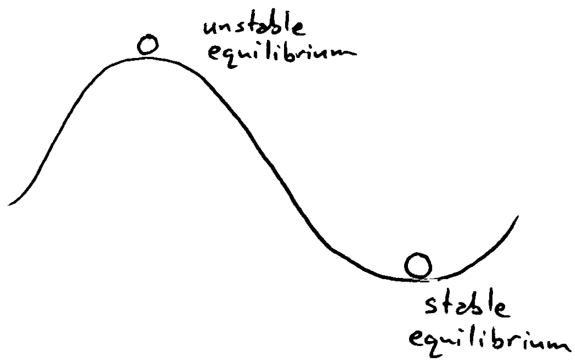
$$\Updownarrow$$

$$F = 0 \qquad \text{(no force!)}$$

If the force is conservative — $F = -\nabla V$ — this is also equivalent to

$$\nabla V(q_0) = 0$$

i.e. we have equilibrium at a critical point of the potential. The equilibrium will be <u>stable</u> if q_0 is a local minimum of V:



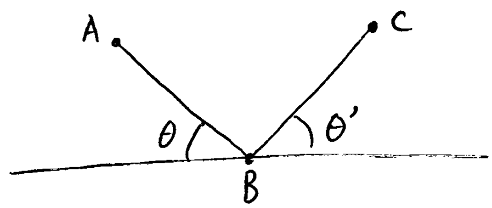
So we have a "Principle of Least Energy" governing stable equilibria. So we have an analogy:

STATICS	DYNAMICS
equilibrium $a = 0$	$F = ma$
potential V	action $S = \int_{t_0}^{t_1} K - V \, dt$
critical points of V	critical points of S

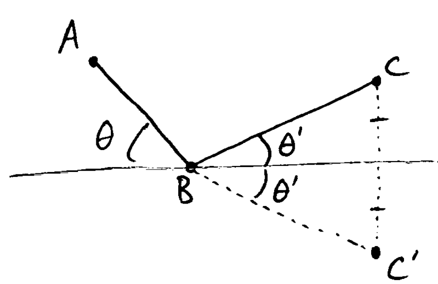
1 April 2005

The Principle of Least Time

Besides the hints from statics, there were also hints from light that nature likes to minimize things. Light in a vacuum moves in straight lines, which minimize distance! But more interestingly, consider a mirror



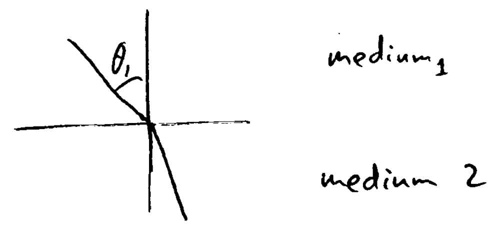
What path does the light choose? Empirical answer: it chooses B such that $\theta = \theta'$: "angle of incidence (θ) equals angle of reflection (θ'). This is precisely the path that minimizes the distance subject to the constraint that it touches the mirror (at at least one point). In fact light takes both the straight path AC & the path ABC. Why does the path ABC minimize the distance (subject to the constraint) when $\theta = \theta'$?



$$
\begin{aligned}
B \text{ minimizes } AB + BC &\iff B \text{ minimizes } AB + BC' \\
&\iff A, B, C' \text{ lie on a line} \\
&\iff \theta = \theta'
\end{aligned}
$$

(note the analogy with method of images in E&M)

The real clue, however, came from the study of refraction:



Snell (& predecessors) noted that each medium has some number n associated to it, the "index of refraction", s.t.

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

(We normalize n s.t. n of a vacuum is 1.)

Someone guessed the solution, i.e. realized that if the speed of light in a medium is proportional to $\frac{1}{n}$, then light will satisfy Snell's law if it's trying to minimize time it takes to get from A to C. The calculation is a bit messy ... but there should be some beautiful proof!

In short, light is not only fast, it's in a hurry! (as if it's trying to keep up its reputation)

How D'Alembert & others guessed the truth.

D'Alembert's principle of virtual work for statics says: equilibrium occurs when

$$F(q_0)\cdot \delta q = 0$$

$\forall\, \delta q \in \mathbb{R}^n$.

D'Alembert generalized this to dynamics by inventing what he called the "inertia force", $-ma$, & said in dynamics, equilibrium occurs when the total force — F + inertia force — vanishes, or

$$(F(q(t)) - ma(t)) \cdot \delta q(t) = 0$$

Now, let's imagine paths

$$q_s(t) = q(t) + s\,\delta q(t)$$

where

$$\delta q(t_0) = \delta q(t_1) = 0$$

& for any function f on the space of paths, ~~call~~ define the variational derivative

$$\delta f = \frac{d}{ds} f(q_s) \Big|_{s=0}$$

Then D'Alembert's principle implies

$$\int_{t_0}^{t_1} (F(q) - m\ddot{q}) \cdot \delta q \; dt = 0$$

for all δq, so if $F = -\nabla V$ we get

$$\int_{t_0}^{t_1} (-\nabla V(q) - m\ddot{q}) \cdot \delta q \; dt = 0$$

$$= \int_{t_0}^{t_1} (-\nabla V(q) \cdot \delta q + m\dot{q}\,\delta\dot{q}) \; dt$$ int. by parts

$$= \int_{t_0}^{t_1} (-\delta V(q) + \frac{m}{2} \delta(\dot{q} \cdot \dot{q})) \; dt$$

So

$$\delta \int_{t_0}^{t_1} (-V + K) \, dt = 0$$

So the path taken is a critical point of

$$S(q) = \int (K - V) \, dt$$

But: what's the real meaning of "inertia force"?
In an accelerating coordinate system, there's a "fictitious force" $-ma$, e.g. centrifugal force (or gravity, though this only begins to make sense at the level of GR)