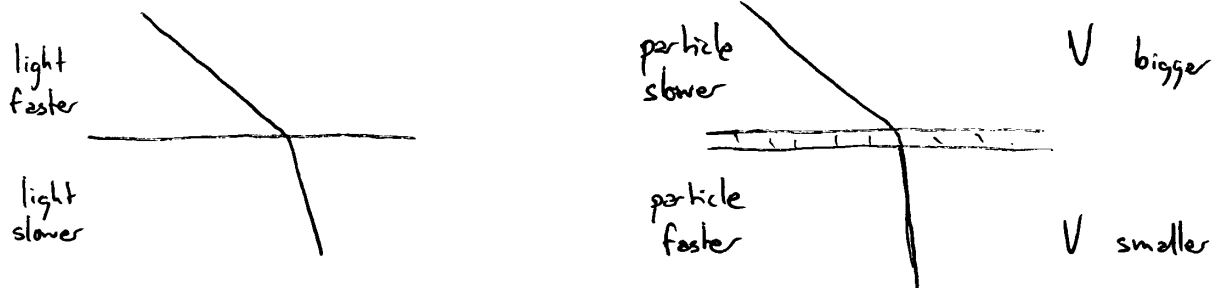# Least Action vs. Least Time

We mentioned Fermat's <u>principle of least time</u> in optics as analogous to the principle of least action in particle mechanics.   This analogy is strange, since in the principle of least action we fix the time interval $q: [0, 1] \longrightarrow Q$.   Also:

light
faster

light
slower

particle
slower

particle
faster

V bigger

V smaller

Nonetheless, Jacobi was able to reinterpret the mechanics of a particle in a potential as an optics problem & "unify" the two minimization principles.   First let's consider light in a medium with a varying index of refraction n (recall $\frac{1}{n} \propto$ speed of light). Suppose it's in $\mathbb{R}^n$ with its usual Euclidean metric. If the light is trying to minimize the <u>time</u>, it's trying to minimize the arclength of its path in the metric

$$g_{ij} = n^2 \delta_{ij}$$

— i.e. index of refraction $n: \mathbb{R}^n \longrightarrow (0, \infty)$ times the usual Euclidean metric

$$\delta_{ij} = \begin{pmatrix} \ddots & & 0 \\ & \ddots & \\ 0 & & \ddots \end{pmatrix}$$

This is just like the free particle in general relativity (minimizing its proper time) except that now g_{ij} is a Riemannian metric:

$$g(v,w) = g_{ij} v^i w^j$$

w. $\quad g(v,v) \geq 0$.

So we'll use the same Lagrangian:

$$L(q,\dot{q}) = \sqrt{g_{ij}(q) \dot{q}^i \dot{q}^j}$$

& get the same Euler-Lagrange equations:

$$\frac{d^2 q^i}{dt^2} + \Gamma^i_{jk} \dot{q}^j \dot{q}^k = 0 \qquad \bigstar$$

if q is parameterized by arclength or more generally $\|\dot{q}\| = \sqrt{g_{ij}\dot{q}^i\dot{q}^j} = $ constant. (where as before the Christoffel symbols Γ are built from derivatives of the metric g). ④ Now, what Jacobi did is show how the motion of a particle in a potential could be viewed as a special case of this. Consider a particle of mass m in Euclidean $\mathbb{R}^n$ with potential $V: \mathbb{R}^n \to \mathbb{R}$. It satisfies $F = ma$, i.e.

$$m \frac{d^2 q^i}{dt^2} = -\partial_i V \qquad \bigstar\bigstar$$

How did Jacobi see $\bigstar\bigstar$ as a special case of $\bigstar$? He considered a particle of energy E & he chose the index of refraction to be

$$n(q) = \sqrt{\frac{2}{m}(E - V(q))}$$

which is just the speed of a particle of energy E when the potential energy is $V(q)$, since

$$\sqrt{\tfrac{2}{m}(E-V)} = \sqrt{\tfrac{2}{m}K} = \sqrt{\tfrac{2}{m}\tfrac{1}{2}m\|\dot{q}\|^2} = \|\dot{q}\|.$$

Note: this is precisely backwards compared to optics, where $n(q)$ is proportional to the reciprocal of the speed of light!!! But let's see that it works:

$$L = \sqrt{g_{ij}(q)\,\dot{q}^i\dot{q}^j}$$

$$= \sqrt{n^2(q)\,\delta_{ij}\,\dot{q}^i\dot{q}^j}$$

$$= \sqrt{\tfrac{2}{m}(E-V(q))\,\dot{q}^2}$$

where $\dot{q}^2 = \dot{q}\cdot\dot{q}$ is just the usual Euclidean dot product $v\cdot w = \delta_{ij}v^iw^j$. We get E-L equations:

$$p_i = \frac{\partial L}{\partial \dot{q}^i} = \sqrt{\tfrac{2}{m}(E-V)}\,\frac{\dot{q}_i}{\|\dot{q}\|}$$

$$F_i = \frac{\partial L}{\partial q^i} = \partial_i\sqrt{\tfrac{2}{m}(E-V(q))}\,\|\dot{q}\|$$

$$= \frac{1}{2}\,\frac{-\tfrac{2}{m}\partial_i V}{\sqrt{\tfrac{2}{m}(E-V)}}\,\|\dot{q}\|$$

$\dot{p} = F$ says:

$$\frac{d}{dt}\sqrt{\tfrac{2}{m}(E-V)}\,\frac{\dot{q}_i}{\|\dot{q}\|} = -\frac{1}{m}\,\partial_i V\,\frac{\|\dot{q}\|}{\sqrt{\tfrac{2}{m}(E-V)}}$$

Jacobi noticed that this is just $F=ma$, or $m\ddot{q}_i = -\partial_i V$, if we

reparameterize q so that $\|\dot{q}\| = \sqrt{\frac{2}{m}(E - V(q))}$ Recall: our Lagrangian gives reparameterization invariant E-L eqns!

27 April 2005

The Ubiquity of Geodesic Motion

We've seen that many classical systems trace out paths that are geodesics: paths $q: [t_0, t_1] \longrightarrow Q$ that are critical points of

$$S(q) = \int_{t_0}^{t_1} \sqrt{g_{ij} \dot{q}^i \dot{q}^j} \, dt$$

which is proper time when (Q, g) is a Lorentzian manifold, or arclength when (Q, g) is a Riemannian manifold:

1) $g(q): T_q Q \times T_q Q \longrightarrow \mathbb{R}$

$$(v, w) \longmapsto g(v, w)$$

is bilinear

2) W.r.t. some basis of $T_q Q$

$$g(v, w) = \delta_{ij} v^i w^j$$

3) $g(q)$ varies smoothly with $q \in Q$.

Lorentzian manifolds represent spacetime, Riemannian manifolds represent space.

We've seen:

1) In the geometric optics approximation, light in $Q = R^n$ acts like particles tracing out geodesics in the metric

$$g_{ij} = n(q)^2 \delta_{ij}$$

where $n: Q \to (0, \infty)$ is index of refraction.

2) Jacobi saw that a particle in $Q = R^n$ in some potential $V: Q \to R$ traces out geodesics in the metric

$$g_{ij} = \frac{2}{m}(E-V) \delta_{ij}$$

if the particle has energy E (where $V < E$).

$V > E \iff$ class, ally forbidden reg...

3) A free particle in general relativity traces out a geodesic on a Lorentzian manifold (Q, g).

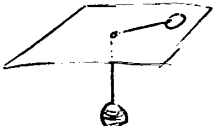In fact all 3 can be generalized to cover every problem we've discussed!

1') Light on any Riemannian manifold (Q, g) with index of refraction $n: Q \to (0, \infty)$ traces out geodesics in the metric $h = n^2 g$.

2') A particle on a Riemannian manifold (Q, g) with potential $V: Q \to R$ trace out geodesics w.r.t. the metric

$$h = \frac{2}{m}(E-V)g$$

if it has energy E. Lots of physical systems can

be described this way :

Atwood machine , , rigid rotating body

$(Q = SO(3))$, tops, etc.

— all of these have Lagrangian which is quadratic fn of velocity + some function of position, so they all fit into this framework.
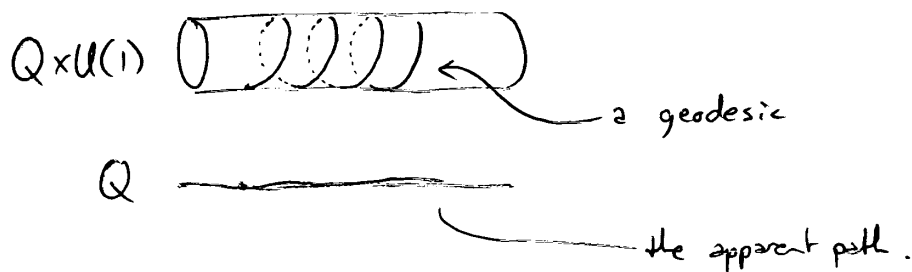
3') <u>Kaluza-Klein theory</u> : a particle with charge e on a Lorentzian manifold (Q, g) in an electromagnetic vector potential follows a path with :

$$\ddot{q}_i = -\Gamma_{ijk} \dot{q}^j \dot{q}^k + \frac{e}{m} F_{ij} \dot{q}^j$$

where

$$F_{ij} = \partial_i A_j - \partial_j A_i .$$

This is actually geodesic motion on $Q \times U(1)$ where $U(1) = \{e^{i\theta} : \theta \in R\}$ is a circle, & $Q \times U(1)$ is given a cleverly designed metric built from g & A

$Q \times U(1)$

a geodesic

Q

the apparent path.

where the amount of "spiralling" — the velocity in $U(1)$ direction is $\frac{e}{m}$. The metric h on $Q \times U(1)$ is built from g & A in a very simple way. Let's pick coordinates x^i on Q where $i \in \{0, ..., n\}$ since we're in $(n+1)$-dim spacetime, & θ is our local coordinate on S^1. The components of h are

$$h_{ij} = g_{ij} + A_i A_j$$

$$h_{i\theta} = h_{\theta i} = -A_i$$

$$h_{\theta\theta} = 1$$

Working out the equations for a geodesic in this metric we get :

$$\ddot{q}_i = -\underbrace{\Gamma_{ijk}}_{\text{Christoffel symbols of } g} \dot{q}^j \dot{q}^k + \frac{e}{m} F_{ij} \dot{q}^j$$

$$\ddot{q}_\theta = 0$$

$$\text{if } \dot{q}_\theta = \frac{e}{m}$$

since F_{ij} is part of the Christoffel symbols for h.

So — every problem we've discussed can be seen as geodesic motion!

29 Apr. 2005

In the Lagrangian approach we focus on the position & velocity of a particle, & compute what the particle does starting from the Lagrangian $L(q, \dot{q})$, which is a function

$$L : TQ \longrightarrow \mathbb{R}$$

where the <u>tangent bundle</u> is the space of position-velocity pairs. But we're led to consider momentum

$$p_i = \frac{\partial L}{\partial \dot{q}^i}$$

since the equations of motion say how it changes:

$$\frac{dp_i}{dt} = \frac{\partial L}{\partial q^i}$$

In the Hamiltonian approach we focus on <u>position</u> & <u>momentum</u>, & compute what the particle does starting from the energy

$$H = p_i \dot{q}^i - L(q, \dot{q})$$

reinterpreted as a function of position & momentum, the <u>Hamiltonian</u>

$$H : T^* Q \longrightarrow \mathbb{R}$$

where the <u>cotangent bundle</u> is the space of position-momentum pairs. In this approach, position & momentum will satisfy <u>Hamilton's equations</u>:

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i} \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i}$$

where the latter is the Euler Lagrange equation

$$\frac{dp_i}{dt} = \frac{\partial L}{\partial q^i}$$
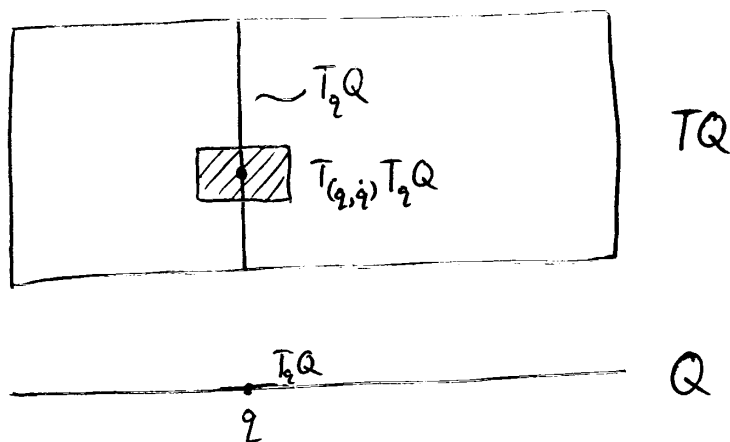
in disguise   (minus sign since $H = p_i \dot{q}^i - L$)

To get to Hamiltonian description, we need to study this map:

$$\lambda : TQ \longrightarrow T^*Q$$
$$(q, \dot{q}) \longmapsto (q, p)$$

where $q \in Q$, $\dot{q}$ is any tangent vector in $T_q Q$ (__not__ the time derivative of something), and p is a cotangent vector in $T_q^* Q := (T_q Q)^*$, given by:

$$p_i = \frac{\partial L}{\partial \dot{q}^i}$$

So: λ is defined using $L : TQ \to \mathbb{R}$. Despite appearances, λ can be defined in a coordinate-free way, as follows.

We want to define "$\frac{\partial L}{\partial \dot{q}^i}$" in a coordinate-free way; it's the "differential of L in the vertical direction" — i.e. the $\dot{q}^i$ directions. We have

$$\pi : TQ \longrightarrow Q$$
$$(q,\dot{q}) \longmapsto q$$

&
$$d\pi : T(TQ) \rightarrow TQ$$

has kernel consisting of **vertical vectors**:

$$VTQ = \ker d\pi \subseteq TTQ.$$

The differential of L at some point $(q,\dot{q}) \in TQ$ is

$$(dL)_{(q,\dot{q})} \in T^*_{(q,\dot{q})} TQ$$

i.e.

$$dL_{(q,\dot{q})} : T_{(q,\dot{q})} TQ \longrightarrow R$$

We can restrict this to $VTQ \subseteq TTQ$, getting
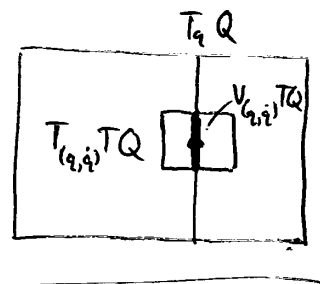
$$f : V_{(q,\dot{q})} TQ \longrightarrow R$$

But note

$$V_{(q,\dot{q})} TQ = T(T_q Q)$$

& since $T_q Q$ is a vector space,

$$T_{(q,\dot{q})} T_q Q \cong T_q Q$$

in a canonical way.

so f gives a linear map

$$p : T_q Q \longrightarrow \mathbb{R}$$

i.e.

$$p \in T_q^* Q \ !$$

This is the momentum.