Conservation Laws for the \(n\)-Body Problem

1. Show that in an \(n\)-body system where the force of the \(j\)-th body on the \(i\)-th body is given by
\[
F_{ij}(t) = f_{ij}(\langle q_i(t) - q_j(t)\rangle) \lambda_{ij}(t)
\]
(where \(\lambda_{ij}\) denotes the unit vector in the direction \(q_i - q_j\) at the time of interest and \(f_{ij} = f_{ji}\); i.e. Newton’s 3rd Law) the energy is conserved.

Note that \(E(t) = T(t) + V(t)\). Let’s work on \(T\) first. We’re going to play fast and loose with notation; hopefully all computations will be clear. Recall that \(T\) is given as \(T = \sum_i (1/2)m_i\dot{q}_i^2\) so that

\[
\frac{dT}{dt} = \sum_i m_i \dot{q}_i \cdot \ddot{q}_i \\
= \sum_i \dot{q}_i \cdot F_i \\
= \sum_i \sum_{i \neq j} \dot{q}_i \cdot F_{ij} \\
= \sum_i \sum_{i \neq j} f_{ij}(\dot{q}_i \cdot \lambda_{ij}) \\
= \sum_i \sum_{i < j} f_{ij}[\dot{q}_i \cdot \lambda_{ij} + \dot{q}_j \cdot \lambda_{ji}] \\
= \sum_i \sum_{i < j} f_{ij}(\dot{q}_i - \dot{q}_j) \cdot \lambda_{ij}.
\]

Now working with \(V\) we have that:

\[
\frac{dV}{dt} = \sum_i \frac{dV_i}{dt} \\
= \sum_i \sum_{i < j} \frac{dV_{ij}}{dt} \\
= -\sum_i \sum_{i < j} f_{ij} \cdot (\dot{q}_i - \dot{q}_j) \cdot \lambda_{ij}
\]

since \(V'_{ij} = -f_{ij}\). Summing the results of these two computations yields \(E'(t) \equiv 0\), and hence energy is conserved.
2. Show that angular momentum is conserved in the $n$-body problem.

\[
\frac{dJ}{dt} = \sum_i J_i(t)
\]

\[
= \sum_i \dot{q}_i \times \dot{p}_i + q_i \times \ddot{p}
\]

\[
= \sum_i m_i (\dot{q}_i \times \dot{q}_i) + q_i \times F_i
\]

\[
= \sum_i \sum_{i \neq j} f_{ij} (q_i \times \lambda_{ij})
\]

\[
= \sum_i \sum_{i<j} f_{ij} (q_i \times \lambda_{ij} + q_j \times \lambda_{ji})
\]

\[
= \sum_i \sum_{i<j} \frac{f_{ij}}{|q_i - q_j|} (q_i \times q_j + q_j \times q_i)
\]

and the last term vanishes since $q_i \times q_j = -q_j \times q_i$. (The second to last equality follows from Newton’s 3rd Law.)