Classical Mechanics Homework
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Conservation of Energy for the n-Body Problem

If Newton's Second Law holds, then energy is conserved.

Solution: The energy is given by $E = T + V$, where T is the kinetic energy and V is the potential energy. If there are n bodies in a system, then $T = \sum_{i=1}^{n} \frac{1}{2} m_i \dot{q}_i^2(t)$ and so $\dot{T} = \sum_{i=1}^{n} m_i \ddot{q}_i \cdot \dot{q}_i = \sum_{i=1}^{n} F_i \cdot \dot{q}_i$, by Newton's Second Law. Since $F_i = \sum_{j \neq i} F_{ij} = \sum_{j \neq i} f_{ij}(|q_i - q_j|) \frac{q_i - q_j}{|q_i - q_j|}$ we have

$$\dot{T} = \sum_{i=1}^{n} \sum_{j \neq i} f_{ij}(|q_i - q_j|) \frac{q_i - q_j}{|q_i - q_j|} \cdot \dot{q}_i = \sum_{i=1}^{n} \left(\sum_{j > i} f_{ij}(|q_i - q_j|) \frac{q_i - q_j}{|q_i - q_j|} \cdot \dot{q}_i + \sum_{j < i} f_{ij}(|q_i - q_j|) \frac{q_i - q_j}{|q_i - q_j|} \cdot \dot{q}_i \right) =$$
$$\sum_{i=1}^{n} \sum_{j > i} \frac{q_i - q_j}{|q_i - q_j|} \cdot (f_{ij}(|q_i - q_j|)\dot{q}_i - f_{ji}(|q_i - q_j|)\dot{q}_j) = \sum_{i=1}^{n} \sum_{j > i} f_{ij}(|q_i - q_j|) \frac{q_i - q_j}{|q_i - q_j|} \cdot (\dot{q}_i - \dot{q}_j),$$

since $f_{ij} = f_{ji}$ by Newton's Third Law. Also $V = \sum_{i=1}^{n} V_i = \sum_{i=1}^{n} \sum_{j > i} V_{ij}(|q_i - q_j|)$ so

$$\dot{V} = \sum_{i=1}^{n} \sum_{j > i} \dot{V}_{ij}(|q_i - q_j|) \frac{d}{dt}(|q_i - q_j|)) = \sum_{i=1}^{n} \sum_{j > i} -f_{ij}(|q_i - q_j|) \frac{q_i - q_j}{|q_i - q_j|} \cdot (\dot{q}_i - \dot{q}_j),$$

since $V'_{ij} = -f_{ij}$. So $\dot{E} = \dot{T} + \dot{V} = 0$ and energy is conserved.

Conservation of Energy for the n-Body Problem

If Newton's Third Law holds, then angular momentum is conserved.

Solution: If there are n bodies in a system, the angular momentum is $J(t) = \sum_{i=1}^{n} J_i(t) = \sum_{i=1}^{n} m_i q_i \times \dot{q}_i$, so

$$\dot{J}(t) = \sum_{i=1}^{n} m_i \dot{q}_i \times \dot{q}_i + q_i \times m \ddot{q}_i = \sum_{i=1}^{n} q_i \times F_i = \sum_{i=1}^{n} q_i \times \left(\sum_{j \neq i} f_{ij}(|q_i - q_j|) \frac{q_i - q_j}{|q_i - q_j|} \right) =$$
$$\sum_{i=1}^{n} \sum_{j \neq i} \frac{f_{ij}(|q_i - q_j|)}{|q_i - q_j|} q_i \times (q_i - q_j) = \sum_{i=1}^{n} \sum_{j \neq i} \frac{f_{ij}(|q_i - q_j|)}{|q_i - q_j|} q_j \times q_i = 0$$

since $q_i \times q_j = -q_j \times q_i$, $f_{ij} = f_{ji}$ and all the terms will cancel.