

Classical Mechanics Homework

February 7, 2008

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This homework gives you some choice in which problems you do, depending on how ambitious you feel. In the section on the Euclidean group you can either do problems 1–3, or if you’re feeling more ambitious, just problem 4. In the section on the Galilei group you can either do problems 5–7, or if you’re feeling more ambitious, just problem 8. Everyone has to do problems 9–10.

The Euclidean Group

Recall that the **orthogonal group** $O(n)$ is the group of linear transformations $R: \mathbb{R}^n \rightarrow \mathbb{R}^n$ that preserve distances: $\|Rx\| = \|x\|$ for all $x \in \mathbb{R}^n$. In other words, $O(n)$ is the group of $n \times n$ matrices with $RR^* = 1$.

Define an element of the **Euclidean group** $E(n)$ to be a pair (R, u) , where $R \in O(n)$ and $u \in \mathbb{R}^n$. Any element (R, u) gives a transformation of n -dimensional Euclidean space built from an orthogonal transformation and a translation:

$$f_{(R,u)}: \mathbb{R}^n \rightarrow \mathbb{R}^n$$

defined by

$$f_{(R,u)}(x) = Rx + u.$$

The map $f_{(R,u)}$ uniquely determines R and u , so we can also think of $E(n)$ as a set of maps.

1. Given two elements $(R, u), (R', u') \in E(n)$ show that

$$f_{(R,u)} \circ f_{(R',u')} = f_{(R'',u'')}$$

for some unique $(R'', u'') \in E(n)$. Work out the explicit formula for (R'', u'') .

This formula lets us define a ‘multiplication’ operation on $E(n)$ by: $(R, u)(R', u') = (R'', u'')$.

2. Given an element $(R, u) \in E(n)$ show that

$$f_{(R,u)}^{-1} = f_{(R',u')}$$

for some unique $(R', u') \in E(n)$. Work out the explicit formula for (R', u') .

This formula lets us define an ‘inverse’ operation on $E(n)$ by: $(R, u)^{-1} = (R', u')$.

3. With multiplication and inverses defined as above, show that $E(n)$ is a group. (Hint: the good way to do this requires almost no calculation.)

Note that as a set we have $E(n) = O(n) \times \mathbb{R}^n$. However, as a group $E(n)$ is not the direct product of the groups $O(n)$ and \mathbb{R}^n , because the formulas for multiplication and inverse are not just

$$(R, u)(R', u') = (RR', u + u'), \quad (R, u)^{-1} = (R^{-1}, -u).$$

Instead, the formulas involve the action of $O(n)$ on \mathbb{R}^n , so we say $E(n)$ is a ‘semidirect’ product of $O(n)$ and \mathbb{R}^n .

4. Suppose that $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a map that preserves distances:

$$|f(x) - f(y)| = |x - y|$$

for all $x, y \in \mathbb{R}^n$. Show that

$$f(x) = Rx + u$$

for some $(R, u) \in E(n)$. Thus we can more elegantly define the Euclidean group to be the group of all distance-preserving transformations of Euclidean space!

The Galilei Group

Define an element of the **Galilei group** $G(n+1)$ to be an triple (f, v, s) where $f \in E(n)$, $v \in \mathbb{R}^n$ and $s \in \mathbb{R}$. We call f a **Euclidean transformation**, v a **Galilei boost** and s a **time translation**.

Any element $(f, v, s) \in G(n+1)$ gives a transformation of $(n+1)$ -dimensional spacetime

$$F_{(f,v,s)}: \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$$

defined by

$$F_{(f,v,s)}(x, t) = (f(x) + vt, t + s)$$

for all $(x, t) \in \mathbb{R}^{n+1}$. The map $F_{(f,v,s)}$ uniquely determines f, v and s , so we can also think of $G(n+1)$ as a set of maps.

5. Given two elements $(f, v, s), (f', v', s') \in G(n+1)$ show that

$$F_{(f,v,s)} \circ F_{(f',v',s')} = F_{(f'',v'',s'')}$$

for some unique $(f'', v'', s'') \in G(n+1)$. Work out the explicit formula for (f'', v'', s'') .

This formula lets us define a ‘multiplication’ operation on $G(n+1)$ by: $(f, v, s)(f', v', s') = (f'', v'', s'')$.

6. Given an element $(f, v, s) \in G(n+1)$ show that

$$F_{(f,v,s)}^{-1} = F_{(f',v',s')}$$

for some unique $(f', v', s') \in G(n+1)$. Work out the explicit formula for (f', v', s') .

This formula lets us define an ‘inverse’ operation on $G(n+1)$ by: $(f, v, s)^{-1} = (f', v', s')$.

7. With multiplication and inverses defined as above, show that $G(n+1)$ is a group. (Again, the good way to do this requires almost no calculation.)

As a set we have $G(n+1) = E(n) \times \mathbb{R}^n \times \mathbb{R}$. However, it is again not the direct product of these groups, but only a semidirect product.

8. Describe some structure on \mathbb{R}^{n+1} such that $G(n+1)$ is precisely the group of all maps $F: \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$ that preserve this structure. Prove that this is indeed the case.

*More generally, we could axiomatically define an $(n+1)$ -dimensional **Galilean spacetime** and prove that the symmetry group of any such thing is isomorphic to $G(n+1)$.*

The Free Particle

Recall that a group G **acts** on a set X if for any $g \in G$ and $x \in X$ we get an element $gx \in X$, and

$$g(g'x) = (gg')(x), \quad 1x = x \tag{1}$$

for all $g, g' \in G$ and $x \in X$. We have just described how the Euclidean group acts on Euclidean space and how the Galilei group acts on Galilean spacetime. Now we will figure out how the Galilei group acts on the phase space of a free particle! Recall that the phase space of a particle in n -dimensional Euclidean space is $X = \mathbb{R}^n \times \mathbb{R}^n$, where a point $(q, p) \in X$ describes the particle's position and momentum. I will tell you how various subgroups of the Galilei group act on X , and you will use that information to figure out how the the whole group acts on X .

- The translation group \mathbb{R}^n is a subgroup of $E(n)$ and thus $G(n+1)$ in an obvious way, and it acts on X as follows:

$$u(q, p) = (q + u, p) \quad u \in \mathbb{R}^n.$$

In other words, to translate a particle we translate its position but leave its momentum alone!

- The orthogonal group $O(n)$ is also a subgroup of $E(n)$ and thus $G(n+1)$ in an obvious way, and it acts on X as follows:

$$R(q, p) = (Rq, Rp) \quad R \in O(n).$$

In other words, to rotate a particle we rotate both its position and momentum!

- The group of Galilei boosts \mathbb{R}^n is a subgroup of $G(n+1)$ in an obvious way, and it acts on X as follows:

$$v(q, p) = (q, p + mv) \quad v \in \mathbb{R}^n.$$

In other words, to boost a particle's velocity by v we add mv to its momentum but leave its position alone!

- Finally, the time translation group \mathbb{R} is a subgroup of $G(n+1)$ in an obvious way, and it acts on X as follows:

$$s(q, p) = (q + sp/m, p) \quad s \in \mathbb{R}.$$

This is where we are assuming the particle is **free**: the force on it is zero, so it moves along at a constant velocity, namely p/m .

9. Assuming that all these group actions fit together to define an action of the whole Galilei group on X , figure out how the whole Galilei group acts on X .

Hint: you'll probably want to use formula (1) and also some results from problems 1–3 and 6–7. An element of the Galilei group is a triple $(f, v, s) \in E(n) \times \mathbb{R}^n \times \mathbb{R}$, but here it's best to think of it as a quadruple $(R, u, v, s) \in O(n) \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, using the fact that $f = (R, u)$. I want you to give me a formula like

$$(R, u, v, s)(q, p) = \dots$$

10. Finally, check that you *really have defined an action* of $G(n+1)$ on X . That is, check equation (1) for all $g = (R, u, v, s)$ and $g' = (R', u', v', s')$ in the Galilei group and all $x = (q, p) \in X$.