Rotations and Angular Momentum

Math 241 Homework
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The goal of this homework is to understand how ***angular momentum generates rotations****. This is true for any system of particles interacting by central forces in $\mathbb{R}^n$, and even more generally — but we'll just consider the case of a particle in $\mathbb{R}^3$. Those of you who are feeling ambitious can skip all the problems except problem 7, which treats a more general case. The rest of you should do problems 1-6.*

First, recall that the **orthogonal group** $\mathrm{O}(n)$ consists of $n \times n$ real matrices R with $RR^* = 1$. (Since we're dealing with real matrices, the adjoint R^* is just the transpose.) These matrices describe operations composed of rotations and reflections. The Lie algebra of $\mathrm{O}(n)$ is denoted $\mathfrak{o}(n)$. We can think of this as a certain vector space of matrices. Let's figure out what this Lie algebra is like, especially for $n = 3$.

The trick is to remember that the Lie algebra is the tangent space at the identity of the Lie group. Thus an $n \times n$ real matrix X lies in $\mathfrak{o}(n)$ iff we can find a curve $R\colon \mathbb{R} \to \mathrm{O}(n)$ going through the identity at $t = 0$:

$$R(0) = 1$$

whose tangent vector at $t = 0$ is X:

$$\dot{R}(0) = X.$$

1. By differentiating the equation $R(t)R(t)^* = 1$, show that the above conditions imply X is **skew-adjoint**:

$$X^* = -X.$$

2. Conversely, given any skew-adjoint $n \times n$ real matrix X, show that

$$e^{tX} = \sum_{n=0} \frac{(tX)^n}{n!}$$

is a curve in $\mathrm{O}(n)$ going through the identity at $t = 0$ whose tangent vector at $t = 0$ is X.

It follows that $\mathfrak{o}(n)$ consists of the $n \times n$ real skew-adjoint matrices!

(By the way, this Lie algebra is precisely the same as $\mathfrak{so}(n)$, the Lie algebra of the rotation group. Including reflections makes $\mathrm{O}(n)$ have two connected components, while the rotation group $\mathrm{SO}(n)$ has one, but their Lie algebras are the same.)

Now let's look at $n = 3$.

3. Show that $\mathfrak{o}(3)$ has a basis given by the matrices

$$X_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, X_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, X_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This implies that any matrix in $\mathfrak{o}(3)$ is of the form

$$\mathbf{a} \cdot \mathbf{X} := a_1 X_1 + a_2 X_2 + a_3 X_3$$

for a unique vector $\mathbf{a} \in \mathbb{R}^3$.

4. Suppose $\mathbf{a} \in \mathbb{R}^3$ is any unit vector. Show that for any vector $\mathbf{v} \in \mathbb{R}^3$

$$(\mathbf{a} \cdot \mathbf{X})\mathbf{v} = \mathbf{a} \times \mathbf{v}.$$

Using this, show that

$$\frac{d}{dt} e^{t\mathbf{a}\cdot\mathbf{X}} \mathbf{v} = \mathbf{a} \times (e^{t\mathbf{a}\cdot\mathbf{X}} \mathbf{v}).$$

This means that as t increases, the vector $e^{t\mathbf{a}\cdot\mathbf{x}}\mathbf{v}$ keeps moving in a direction perpendicular to itself and also $\mathbf{a}$. You can also work out the speed at which it moves. Use these ideas to show the matrix $e^{t\mathbf{a}\cdot\mathbf{X}}$ describes a counterclockwise rotation by the angle t around the axis pointing in the direction $\mathbf{a}$.

Now let's consider a particle in $\mathbb{R}^3$. Its phase space is $X = \mathbb{R}^3 \times \mathbb{R}^3$, a Poisson manifold with

$$\{F, G\} = \sum_{i=1}^{3} \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i} - \frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i}.$$

In the previous homework we saw how rotations/reflections act on this phase space: for any $R \in \mathrm{O}(3)$ and any point $(\mathbf{q}, \mathbf{p}) \in X$, we have

$$R(\mathbf{q}, \mathbf{p}) = (R\mathbf{q}, R\mathbf{p}).$$

Earlier in class we saw that the angular momentum of a particle is given by

$$\mathbf{J} = \mathbf{q} \times \mathbf{p}.$$

Now we will finally see how these ideas are related!

For any unit vector $\mathbf{a} \in \mathbb{R}^3$, let $F = \mathbf{a} \cdot \mathbf{J}$. This defines a function $F \in C^\infty(X)$ called the **angular momentum around the a axis**, or more sloppily the 'angular momentum in the $\mathbf{a}$ direction'.

5. Calculate the vector field $v_F \in \mathrm{Vect}(X)$.

6. Show that

$$\exp(t v_F)(q, p) = e^{t\mathbf{a}\cdot\mathbf{X}}(q, p) \tag{1}$$

for all $(q, p) \in X$.

In other words, ***angular momentum around the* a *axis generates a 1-parameter group whose action on phase space is precisely rotation about the* a *axis.*** So, in the great dictionary relating symmetries and conserved quantities, rotation symmetry goes along with conservation of angular momentum.

7. State and prove a version of equation (1) for a free particle in $\mathbb{R}^n$. Hint: in n dimensions, we can no longer speak of rotations about an axis, nor angular momentum around an axis.