1 The derivative of the equation $R(t)R(t)^* = 1$ is $R(t)R'(t)^* + R'(t)R(t)^* = 0$. Evaluating this at $t = 0$ gives $1X^* + X1 = 0$, or $X^* = -X$.

2

$$\left(e^{tX}\right)^* = \sum_{n=0} \frac{(tX^*)^n}{n!} = \sum_{n=0} \frac{(-tX)^n}{n!} = e^{-tX}.$$

Since tX and $-tX$ commute, I get $e^{tX}\left(e^{tX}\right)^* = e^{tX}e^{-tX} = e^{tX-tX} = e^0 = 1$. Thus, $e^{tX} \in \mathrm{O}(n)$. Of course, $e^{0X} = 1$. Differentiating termwise, as is allowed for entire functions on Banach algebras,

$$\frac{d}{dt}e^{tX} = \sum_{n=0} \frac{nt^{n-1}X^n}{n!} = \sum_{n=1} \frac{(tX)^{n-1}X}{(n-1)!} = X\sum_{n=0} \frac{(tX)^n}{n!} = Xe^{tX},$$

which at $t = 0$ is $X1 = X$.

3 If $X \in \mathfrak{o}(n)$, then $X_{i,j} = -X_{j,i}$ for indices i, j. If $i = j$, it follows that $X_{i,j} = 0$. If $i > j$, then $X_{i,j}$ is determined by $X_{j,i}$, where $j < i$. Thus, X is determined by $X_{i,j}$ for $i < j$. Conversely, given any values for $X_{i,j}$ for $i < j$, let $X_{i,j}$ be $-X_{j,i}$ when $i > j$ and let $X_{i,j}$ be 0 when $i = j$. Then $X \in \mathfrak{o}(n)$. Thus, a basis of $\mathfrak{o}(n)$ is $\{E_{i,j} \vdots\, i < j\}$, where $\left(E_{i,j}\right)_{i,j} = 1$, $\left(E_{i,j}\right)_{j,i} = -1$, and every other component of $E_{i,j}$ is 0. (That is, $\left(E_{i,j}\right)_{k,l} := \delta^{i,j}_{k,l}$.) When $n = 3$, we have $\{E_{1,2}, E_{1,3}, E_{2,3}\}$. Since $X_1 = -E_{2,3}$, $X_2 = E_{1,3}$, and $X_3 = -E_{1,2}$, another basis is $\{X_1, X_2, X_3\}$.

4 If $\mathbf{v} = (v_1, v_2, v_3)$, then

$$(\mathbf{a}\cdot\mathbf{X})\mathbf{v} = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} a_2v_3 - a_3v_2 \\ a_3v_1 - a_1v_3 \\ a_1v_2 - a_2v_1 \end{pmatrix} = \mathbf{a}\times\mathbf{v}.$$

Thus, the t derivative of $e^{t\mathbf{a}\cdot\mathbf{X}}\mathbf{v}$ is $(\mathbf{a}\cdot\mathbf{X})e^{t\mathbf{a}\cdot\mathbf{X}}\mathbf{v} = \mathbf{a}\times e^{t\mathbf{a}\cdot\mathbf{X}}\mathbf{v}$.

5 Let G be any smooth function on the manifold X. Then

$$\begin{aligned} v_F[G] = \{F, G\} &= \frac{\partial F}{\partial p_1}\frac{\partial G}{\partial q_1} - \frac{\partial F}{\partial q_1}\frac{\partial G}{\partial p_1} + \frac{\partial F}{\partial p_2}\frac{\partial G}{\partial q_2} - \frac{\partial F}{\partial q_2}\frac{\partial G}{\partial p_2} + \frac{\partial F}{\partial p_3}\frac{\partial G}{\partial q_3} - \frac{\partial F}{\partial q_3}\frac{\partial G}{\partial p_3} \\ &= (a_2q_3 - a_3q_2)\frac{\partial G}{\partial q_1} - (a_3p_2 - a_2p_3)\frac{\partial G}{\partial p_1} + (a_3q_1 - a_1q_3)\frac{\partial G}{\partial q_2} \\ &\quad - (a_1p_3 - a_3p_1)\frac{\partial G}{\partial p_2} + (a_1q_2 - a_2q_1)\frac{\partial G}{\partial q_3} - (a_2p_1 - a_1p_2)\frac{\partial G}{\partial p_3} \\ &= (\mathbf{a}\times\mathbf{q})\cdot\nabla_{\mathbf{q}}G + (\mathbf{a}\times\mathbf{p})\cdot\nabla_{\mathbf{p}}G. \end{aligned}$$

Thus, $v_F = (\mathbf{a}\times\mathbf{q})\cdot\nabla_{\mathbf{q}} + (\mathbf{a}\times\mathbf{p})\cdot\nabla_{\mathbf{p}}$. Identifying the tangent spaces to $\mathbb{R}^3\times\mathbb{R}^3$ with $\mathbb{R}^3\times\mathbb{R}^3$ itself, this becomes $v_F = (\mathbf{a}\times\mathbf{q}, \mathbf{a}\times\mathbf{p}) = (\mathbf{a}, \mathbf{a})\,(\times, \times)\,(\mathbf{q}, \mathbf{p})$.

6 At $t = 0$, both sides of equation (1) reduce to $(\mathbf{q}, \mathbf{p})$. The t derivative of the left side is $v_F|_{\exp(tv_F)(\mathbf{q},\mathbf{p})} = (\mathbf{a}, \mathbf{a})\,(\times, \times)\exp(tv_F)(\mathbf{q}, \mathbf{p})$; the t derivative of the right side is $(\mathbf{a}\times e^{t\mathbf{a}\cdot\mathbf{X}}\mathbf{q}, \mathbf{a}\times e^{t\mathbf{a}\cdot\mathbf{X}}\mathbf{p}) = (\mathbf{a}, \mathbf{a})\,(\times, \times)\, e^{t\mathbf{a}\cdot\mathbf{X}}(\mathbf{q}, \mathbf{p})$. Thus, both sides satisfy the same initial value problem, so they are equal.

7 Recall that we have the basis $\{E_{i,j} \vdots i < j\}$ for $\mathfrak{o}(n)$. Then an element X of $\mathfrak{o}(n)$ is given uniquely by a 2vector $\mathbf{a}$ in $\Lambda^2\mathbb{R}^n$, under $X = \sum_{i<j} a_{i,j}E_{i,j}$. Symbolically, I'll indicate this as $X_\mathbf{a}$. Then $X_\mathbf{a}\mathbf{v} = \mathbf{a}\cdot\mathbf{v}$, where $\mathbf{a}$ is treated as an antisymmetric dyadic for purposes of the dot product. The t derivative of $e^{tX_\mathbf{a}}\mathbf{v}$ is $X_\mathbf{a}e^{tX_\mathbf{a}}\mathbf{v} = \mathbf{a}\cdot e^{tX_\mathbf{a}}v$.

If $(\mathbf{q},\mathbf{p}) \in \mathbb{R}^n \times \mathbb{R}^n$, then let $\mathbf{J}$ be the 2vector $\mathbf{q}\wedge\mathbf{p}$. Then let F be the normalised dot product $\mathbf{a}\cdot\mathbf{J}$. That is, $F = \sum_{i<j} a_{i,j}(q_ip_j - q_jp_i)$.

Given $G \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$,

$$\begin{aligned}
v_F[G] = \{F,G\} &= \sum_k \left(\frac{\partial F}{\partial p_k}\frac{\partial G}{\partial q_k} - \frac{\partial F}{\partial q_k}\frac{\partial G}{\partial p_k}\right) = \sum_k\sum_{i<j}\left(a_{i,j}(q_i\delta_{j,k} - q_j\delta_{i,k})\frac{\partial G}{\partial q_k} - a_{i,j}(\delta_{i,k}p_j - \delta_{j,k}p_i)\frac{\partial G}{\partial p_k}\right) \\
&= \sum_{i<j} a_{i,j}\left(q_i\frac{\partial G}{\partial q_j} - q_j\frac{\partial G}{\partial q_i} + p_i\frac{\partial G}{\partial p_j} - p_j\frac{\partial G}{\partial p_i}\right) = \sum_{i,j} a_{i,j}\left(q_i\frac{\partial G}{\partial q_j} + p_i\frac{\partial G}{\partial p_j}\right) \\
&= (\mathbf{a}\cdot\mathbf{q})\cdot\nabla_\mathbf{q}G + (\mathbf{a}\cdot\mathbf{p})\cdot\nabla_\mathbf{p}G.
\end{aligned}$$

Thus, $v_F = (\mathbf{a}\cdot\mathbf{q})\cdot\nabla_\mathbf{q} + (\mathbf{a}\cdot\mathbf{p})\cdot\nabla_\mathbf{p} = (\mathbf{a}\cdot\mathbf{q},\mathbf{a}\cdot\mathbf{p}) = (\mathbf{a},\mathbf{a})\,(\cdot,\cdot)\,(\mathbf{q},\mathbf{p})$.

In this context, equation (1) becomes

$$\exp(tv_F)(\mathbf{q},\mathbf{p}) = e^{tX_\mathbf{a}}(\mathbf{q},\mathbf{p}).$$

Again, at $t = 0$, both sides become $(\mathbf{q},\mathbf{p})$. The t derivative of the left side is $(\mathbf{a},\mathbf{a})\,(\cdot,\cdot)\exp(tv_F)(\mathbf{q},\mathbf{p})$, and that of the right side is $(\mathbf{a},\mathbf{a})\,(\cdot,\cdot)\,e^{tX_\mathbf{a}}(\mathbf{q},\mathbf{p})$. Therefore, the two sides are still equal.