The Euclidean Group

1. \( f_{R,u}(f_{R,u}(x)) = f_{R,u}((R'x + u') = R(R'x + u') + u = (RR')x + (Ru' + u) = f_{RR,Ru'+u}(x) \), so \((R',u') = (RR',Ru' + u)\) will work and is uniquely determined by the composed function.

2. If \( x' = f_{R,u}(x) = Rx + u \), then \( x = R^{-1}(x' - u) = R^{-1}x' + (-R^{-1}u) = f_{R^{-1},-R^{-1}u}(x) \), so \((R',u') = (R^{-1},-R^{-1}u)\) will work and is uniquely determined by the composed function.

3. There is of course a group of all invertible functions on \( \mathbb{R}^n \), the permutation group \( \mathbb{R}^n! \). The previous problems prove that \( E(n) \) is a subset of \( \mathbb{R}^n! \) that is closed under the operations of composition and inversion. Furthermore, \( f_{1,0}(x) = 1x + 0 = x \), so \( E(n) \) also owns the identity function. Therefore, \( E(n) \) is a subgroup of \( \mathbb{R}^n! \), in particular a group.

4. \( G \) is constructed as the range of a function from \( \mathbb{R}^n \) to \( E(n) \), which is invertible by the fundamental property of ordered pairs. First, \( (1,u)(1,u') = (11,1u' + u) = (1,u + u') \), so \( G \) is closed under multiplication and the correspondence between \( G \) and \( \mathbb{R}^n \) preserves that operation. Next, \( (1,u)^{-1} = (1^{-1},-1^{-1}u) = (1,-u) \), so \( G \) is closed under inverses and the correspondence between \( G \) and \( \mathbb{R}^n \) preserves that operation. Finally, the identity element of \( E(n) \) is \((1,0)\), so \( G \) owns the identity element and it corresponds to the identity element \( 0 \) of \( \mathbb{R}^n \). Therefore, \( G \) is a subgroup of \( E(n) \) that is isomorphic to \( \mathbb{R}^n \). Also, \((R,u) \times (1,u)(R,u)^{-1} = (R1,Ru + u')(R^{-1},-R^{-1}u) = (RR^{-1},-RR^{-1}u' + Ru + u') = (1,Ru)\), which belongs to \( G \), so the subgroup \( G \) is normal.

\( H \) is constructed as the range of a function from \( O(n) \) to \( E(n) \), which is invertible by the fundamental property of ordered pairs. First, \( (R,0)(R',0) = (RR',R0 + 0) = (RR',0) \), so \( H \) is closed under multiplication and the correspondence between \( H \) and \( O(n) \) preserves that operation. Next, \( (R,0)^{-1} = (R^{-1},-R^{-1}0) = (R^{-1},0) \), so \( H \) is closed under inverses and the correspondence between \( H \) and \( O(n) \) preserves that operation. Finally, the identity element of \( E(n) \) is \((1,0)\), so \( H \) owns the identity element and it corresponds to the identity element \( 1 \) of \( O(n) \). Therefore, \( H \) is a subgroup of \( E(n) \) that is isomorphic to \( O(n) \).

\((1,u)(R,0) = (1R,10 + u) = (R,u)\), so by the fundamental property of ordered pairs, every element of \( E(n) \) is a unique product of an element of \( G \) and an element of \( H \).

5. Let \( u = f(0) \), and let \( Rx \) be \( f(x) - u \) for \( x \in \mathbb{R}^n \). Note that \( R0 = u - u = 0 \) and

\[ |Rx - Ry| = |(f(x) - u) - (f(y) - u)| = |f(x) - f(y)| = |x - y|, \]

so \( R \) preserves the origin and lengths. Therefore, \( R \) must be an element of \( O(n) \). Since \( f(x) = Rx + u \), the desired result follows.

The Galilei Group

6. Set \( f = f_{R,u} \) and \( f' = f_{R',u'} \). Then

\[ F_{f,v,s}(F_{f',v',s'}(x,t)) = F_{f,v,s}(R'R'x + u' + v't, t + s') = (R(R'x + u' + v't) + u + v(t + s'), t + s' + s) = ((RR')x + (Ru' + u + vs') + (Rs' + v)t, t + (s' + s)) = F_{RR',Rus'+vs'+Rs'+v,s'+s}, \]

so \((f'',u'',s'') = (f_{RR',Rus'+vs'+Rs'+v},Rus'+v,s'+s)\) will work and is uniquely determined by the composed function.

7. Set \( f = f_{R,u} \). Then if \((x',t') = F_{f',v',s}(x,t) = (f(x) + vt, t + s) = (Rx + u + vt, t + s),\) then \( t = t' - s = t' - (-s) \) and \( x = R^{-1}(x' - u - vt) = R^{-1}(x' - u - v(t' - s)) = R^{-1}x' + (sR^{-1}v - R^{-1}u) + (-R^{-1}v)t'\), so \((x,t) = F_{f_{R^{-1},sR^{-1}v-R^{-1}u,-R^{-1}v,-s}}(f',v',s') = (f'_{R^{-1},sR^{-1}v-R^{-1}u,-R^{-1}v,-s})\) will work and is uniquely determined by the composed function.

8. There is of course a group of all invertible functions on \( \mathbb{R}^{n+1} \), the permutation group \( \mathbb{R}^{n+1!} \). The previous problems prove that \( G(n+1) \) is a subset of \( \mathbb{R}^{n+1!} \) that is closed under the operations of composition and inversion. Furthermore, \( f_{1,0,0}(x,t) = (x + 0t, t + 0) = (x,t) \), so \( G(n+1) \) also owns the identity function. Therefore, \( G(n+1) \) is a subgroup of \( \mathbb{R}^{n+1!} \), in particular a group.
Given points \( p, q, r \in \mathbb{R}^{n+1} \), define vectors \( v = q - p \) and \( w = r - p \) tangent to \( \mathbb{R}^{n+1} \) at \( p \). Write \( v = (v_x, v_t) \)
and \( w = (w_x, w_t) \). I claim that Galilean transformations preserve the time difference \( w_t - v_t \)
and the relative speed \( |w_x/w_t - v_x/v_t| \). Now, the latter is not defined if \( v_t \) or \( w_t \) is 0, but since these denominators are also time differences, \( |v_xw_x - w_xv_x| \) should be preserved as well. Furthermore, I claim that any transformation that preserves these is Galilean. Thus in terms of the original points \( p = (p_x, p_t), q = (q_x, q_t), \) and \( r = (r_x, r_t) \), the claimed complete invariant is the ordered pair \((q_rq_x - p_rq_x - r_tq_x + p_tq_x + r_tp_x - q_tp_x, r_t - q_t)\).

First, let me verify that this is an invariant. Applying \( F_{f,v,s} \), where \( f = f(x,t) \), the first component of the claimed invariant becomes
\[
[q_t + s)(Rr_x + u + vr_t) – (p_t + s)(Rr_x + u + vr_t) - (r_t + s)(Rr_x + u + vr_t)
+ (p_t + s)(Rr_x + u + vr_t) + (r_t + s)(Rr_x + u + vr_t) - (q_t + s)(Rr_x + u + vr_t)]
= [qtRr_x - prRr_x - rtRr_x + prRr_x + rtRr_x - qrRr_x]
= [q_t r_x - p_t r_x - r_t q_x + p_t q_x + r_t p_x - q_t p_x],
\]
since \( R \in O(n) \) is linear and preserves lengths. Meanwhile, the second component becomes \((r_t + s) - (q_t + s) = r_t - q_t \). Thus, this is indeed an invariant.

Now suppose that \( F: \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1} \) preserves this invariant. Let \( s \) be the second component of \( F(0,0) \).
Setting \( q = (0,0) \) and \( r = (x,t) \), I see from the second component of the invariant that \( t + s \) is the second component of \( F(x,t) \). Thus let \( F(x,t) \) be the first component of \( F(x,t) \), so that \( F(x,t) = (F_x(x,t),t+s) \).
Now let \( u \) be \( F_x(0,0) \), let \( v = F_x(1,0) - u \), and let \( R \) be \( F_x(x,1) - u - v \) for \( x \in \mathbb{R}^n \). Note that \( R0 = (v + u) - u - v = 0 \). Setting \( p = (0,0), q = (y,1), \) and \( r = (x,1) \),
\[
|R_x - R_y| = |F_x(x,1) - u - v| - |F_x(y,1) - u - v| = |F_x(x,1) - F_x(y,1)|
= |(1 + s)F_x(x,1) - sF_x(x,1) - (1 + s)F_x(y,1) + sF_x(y,1)|
= (1 + s)|F_x(x,1) - F_x(y,1)| = |F_x(x,1) - F_x(y,1)|
= |F_x(x,t) - tR(x/t - t u - u)|. Since every element of \( O(n) \) is linear, it follows that \( F_x(x,t) \) must be \((Rx + u + vt, t + s) \). Thus \( F = F_{j,r,u,v,s} \in G(n + 1) \).

The Free Particle
Since our Galilean transformations have been passive, not active, translation acts as \( s(x,p) = (q - sp/m, p) \)
on \( (q + sp/m, p) \).

10 In \( G(n + 1), (f,v,s) = (1, v, s)(f, 0, 0) = (1, 0, s)(1, v, 0)(f, 0, 0) \), while I already know that \( (R, u, v, s) = (1, u, v, s) \) for \( (0, 0) \) in \( E(n) \). That is, \( (R, u, v, s) = (1, 0, s)(1, 0, 0)(1, u, v, 0)(1, u, v, 0)(R, 0, 0, 0) \). Then
\[
(R, u, v, s)(p, q, p) = (1, 0, s)(1, 0, 0)(1, u, 0, 0)(R, 0, 0, 0)(q, p) = (1, 0, s)(1, 0, 0)(1, u, 0, 0)(Rq, Rp)
= (1, 0, s)(1, 0, 0)(Rq + u, Rp) = (1, 0, s)(Rq + u, Rp) + mv
= (Rq + u - s(Rp + mv)/m, Rp + mv) = (Rq + u - sRp/m - sv, Rp + mv).
\]

11
\[
(R, u, v, s)(R, u', v', s')(q, p) = (R, u, v, s)(R'q + u' - s'R'p/m - s'v', R'p + mv')
= (R'R'q + u' - s'R'p/m - s'v') + u - sR(Rp + mv')/m - sv, R(R'p + mv') + mv
= (RR'Rq + Ru' - sRR'p/m - sRv' + u - sRR'p/m - sRv' - sv, RR'Rp + mRv' + mv)
= (RR'Rq + Ru' + u + s v - sRR'p/m - sRv' - sRv' - sv, RR'Rp + mRv' + mv)
= (RR'Rq + (Ru' + u + s v - (s + s)(RR'Rp/m - (s + s)(Rv' + v), (RR'Rp + mRv' + v))
= (RR'R, Ru' + u + s v, Rv' + v, s' + s)(q, p) = ((R, u, v, s)(R', u', v', s'))(q, p).
\]

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Also, if \((q', p') = (R, u, v, s)(q, p) = (Rq + u - sRp/m - sv, Rp + mv)\), then \(p = R^{-1}(p' - mv) = R^{-1}p' - mR^{-1}v = R^{-1}p' + m(-R^{-1}v)\) and

\[
q = R^{-1}(q' - u + sRp/m + sv) = R^{-1}(q' - u + sR(R^{-1}p' - mR^{-1}v)/m + sv)
\]

\[
= R^{-1}q' - R^{-1}u + sR^{-1}p'/m - sv + sv = R^{-1}q' + sR^{-1}v - R^{-1}u + sR^{-1}p'/m - sR^{-1}v
\]

\[
= R^{-1}q' + (sR^{-1}v - R^{-1}u) - (-s)R^{-1}p'/m - (-s)(-R^{-1}v),
\]

so \((q, p) = (R^{-1}, R^{-1}sv - R^{-1}u, -R^{-1}v, -s)(q', p') = (R, u, v, s)^{-1}(q', p')\). Finally,

\[(1, 0, 0, 0)(q, p) = (1q + 0 - 0 \cdot 1p/m - 0 \cdot 0, 1p + m0) = (q, p).
\]

Therefore, this is an action of \(G(n + 1)\) on \(\mathbb{R}^{2n}\).