Classical Mechanics, Homework 5

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Solution to 4:

Given the map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ and the condition that,

(1) $\quad |f(x) - f(y)| = |x - y|,$

we want f to be a smooth bijection. In what follows we assume that there exists a sequence $\{a_n\} \mid f(x) = \sum_{n=0}^{\infty} a_n x^n$, that f is smooth means we have a chance of identifying the sequence a_n with the n-derivatives of f denoted $f^{(n)}$ and indeed this will be the case. However, the a_n need not be a sequence of scalars and in particular for $n \neq m$, $a_n + a_m$ need not be defined. The isometry in (1) requires that the a_n for $n \geq 2$ are identically zero (or the zero matrix etc.) for since:

$$\left| \sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty} a_n y^n \right| = |x - y|,$$

can happen only for $n < 2$ as all higher power terms must cancel with $x \neq y$. Thus far we have:

$$f(x) = a_0 + a_1 x.$$

Now because $f(x)$ and x are to be vectors in the vector space $\mathbb{R}^n$, closure, under regular addition between elements of a vector space, implies that $a_0 \in \mathbb{R}^n$ and the product $a_1 x \in \mathbb{R}^n$ necessarily. The proof is trivial first take the map f_0 with $a_1 \equiv 0$ then take the map f_1 with $a_0 \equiv 0$ and recall closure of vector spaces under addition. Hence we can identity a_0 with u. Now the above together with the fact that we want f to be invertable, i.e. $f \circ f^{-1} = f^{-1} \circ f = x$, with the inverse given formally by the expression, $f^{-1}(x) = a_1^{-1} x - a_1^{-1} a_0$, means that a_1 must be itself a bijective linear transformation $a_1\colon \mathbb{R}^n \to \mathbb{R}^n$. Hence a_1 is at least an element of the group of invertable, real, $n \times n$ matrices, i.e. $\mathrm{GL}(n, \mathbb{R}^n)$. To utilise the isometry to its fullest we recall the meaning of the operation $|\cdot|$ and invoke the Euclidean metric defined as follows:

(2) $\quad \langle x,\, y\rangle := \sum_{i=1}^{n} x_i y_i \quad \forall x, y \in \mathbb{R}^n.$

However, there is a neat identity that relates $|\cdot|$ to (2), as follows:

(3) $\quad \langle x,\, y\rangle = \frac{1}{4}(\, |x+y|^2 - |x-y|^2\,).$

The isometric map f_1 implies that a_1 respects the isometry relation by itself. This means we can apply a_1 to both sides of (3) and by linearity of a_1's actions on elements of $\mathbb{R}^n$ we get:

$$\begin{aligned} \langle a_1 x,\, a_1 y\rangle &= \frac{1}{4}(\, |a_1 x + a_1 y|^2 - |a_1 x - a_1 y|^2\,) \\ &= \frac{1}{4}(\, |a_1(x+y)|^2 - |a_1(x-y)|^2\,) \qquad \text{linearity} \\ &= \frac{1}{4}(\, |x+y|^2 - |x-y|^2\,) \qquad \text{isometry.} \end{aligned}$$

But then we can immediately read off:

$$(4) \qquad \begin{aligned} \langle a_1 x,\, a_1 y\rangle &= \langle x,\, a_1^{\dagger} a_1 y\rangle \\ &= \langle x,\, y\rangle \end{aligned}$$

Hence $a_1^\dagger = a_1^{-1}$ but on the real euclidean space $\mathbb{R}^n$ we have that $a_1^\dagger = a_1^T$ where a_1^T denotes the transpose and therefore we can make the indentification, $a_1 = R$ with $R \in \mathrm{O}(n)$. Consequently, we have shown that for some pair $(R, u) \in E(n)$ the isometry preserving map $f(x) \mid \quad |f(x) - f(y)| = |x - y|$ can be expressed as:

$$f(x) = Rx + u$$

Solution to 5:

Any element of the Galilei Group defined here as a triple (f, v, s) with $f \in E(n), v \in \mathbb{R}^n$ and $s \in \mathbb{R}$ gives a transformation of (n+1) dimensional spacetime as:

$$F_{(f,v,s)} \colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}.$$

The defined action of the map $F_{(f,v,s)}$ on a spacetime pair (x, t) can be defined by:

$$F_{(f,v,s)}(x, t) = (f(x) + vt, t + s)$$

For any two elements $F_{(f,v,s)}$ and $F'_{(f',v',s')}$ heretofore simply F and F' respectively we want an explicit formula for the composite action $F \circ F' := F''$ of F'' on a spacetime pair (x, t). From the definition of how F acts on a spacetime pair we can write unambiguously the composite action as follows:

$$\begin{aligned}
(F \circ F')(x, t) &= F(f'(x) + v't, t + s') \\
&= F(R'x + u' + v't, t + s') \text{ action of } f \in E(n) \text{ on } x \\
&= (f(R'x + u' + v't) + v(t + s'), t + s' + s) \\
&= (R(R'x + u' + v't) + u + v(t + s'), t + s' + s) \\
&= (RR'x + Ru' + Rv't + u + vt + vs', t + s' + s) \qquad (5) \\
&\equiv F''(x, t)
\end{aligned}$$

Now enters ambiguity; however, this is good. We are free to define consecutive additive actions in a manner which ensures that the defined action is self-consistent. Hence if we make the following defining identifications,

$$\begin{aligned}
(6) \qquad RR' &:= R'' \\
(7) \qquad Ru' + u + vs' &:= u'' \text{and consequently:} \\
(8) \qquad f''(x) &:= RR'x + Ru' + u + vs' := R''x + u'' \\
(9) \qquad Rv' + v &:= v'' \\
(10) \qquad s + s' &:= s'',
\end{aligned}$$

equation (5) can now be written as:

$$\begin{aligned}
(RR'x + Ru' + Rv't + u + vt + vs', t + s' + s) &= (R''x + u'' + v''t, t + s'') \\
&= (f''(x) + v''t, t + s'') \\
&\equiv F''(x, t)
\end{aligned}$$

This defines a self-consistent composition acting on a spacetime pair (x, t) in accordance with the given definition from the onset.

Solution to 6:

Next, we would like the notion of an inverse action F^{-1}. In the set of identifications (6)-(10), if we instead make the alternative identifications,

$$\begin{array}{llllll} R & := & R'^{-}1 & \Longrightarrow & R'' & = & \mathbb{I} \\ u & := & R'^{-}1v's' - R'^{-}1u' & \Longrightarrow & u'' & = & 0 \\ v & := & -R'^{-}1v' & \Longrightarrow & v'' & = & 0 \\ s & := & -s' & \Longrightarrow & s'' & = & 0, \end{array} \tag{11}$$

then $F \equiv F^{-1}$ results in $(F \circ F')(x,t) = (F^{-1} \circ F')(x,t) = (x,t)$.

Solution to 7:

Lastly, we can now show this defines a group. Recalling that the definition of an algebraic group is a set G, which exhibits under some operation $*$ (taking in two arguments a, b and returning one, c) the properties of closure ($\forall\ a, b \in G \Longrightarrow c \in G$), associativity ($\forall\ a, b, c \in G, (a*b)*c = a*(b*c)$) the existence of an identity ($\mathbb{I} \mid \quad a\mathbb{I} = \mathbb{I}a = a$) and the existence of an inverse ($a^{-1} \mid \quad aa^{-1} = a^{-1}a = \mathbb{I}$). Though in general composition of maps is noncommutative, we want our identity to be well behaved so we check that $F^{-1} \circ F = F \circ F^{-1}$. Since we have already calculated $(F^{-1} \circ F')(x,t) = (x,t)$ we need only show that $(F' \circ F^{-1})(x,t) = (x,t)$.

$$\begin{aligned} (F' \circ F^{-}1)(x,t) &= F'(f^{-1}(x) + v^{-1}t, t + s^{-1}) \\ &= F'(R^{-1}x + u^{-1} + v^{-1}t, t + s^{-1}) \quad \text{symbolically} \\ &= F'(R'^{-1}x + (R'^{-1}v's' - R'^{-1}u') - R'^{-1}v't, t - s') \quad \text{explicitly} \\ &= (f'(R'^{-1}x + R'^{-1}v's' - R'^{-1}u' - R'^{-1}v't) + v'(t - s'), t - s' + s') \\ &= (R'(R'^{-1}x + R'^{-1}v's' - R'^{-1}u' - R'^{-1}v't) + u' + v't - v's', t) \\ &= (R'R'^{-1}x + R'R'^{-1}v's' - R'R'^{-1}u' - R'R'^{-1}v't + u' + v't - v's', t) \\ &= (x + v's' - u' - v't + u' + v't - v's', t) \\ &= (x + v's' - v's' + v't - v't + u' - u', t) \\ &= (x,t) \end{aligned}$$

Hence, our set has an inverse. The associativity of composition ensures that our maps as defined are associative. The identity element is found immediately. It is simply the object $F_{\mathbb{I}}(x,t) := (f_{\mathbb{I}}(x) + v_{\mathbb{I}}t,\ t + s_{\mathbb{I}})$, where $f_{\mathbb{I}}(x) := R_{\mathbb{I}}x + u_{\mathbb{I}}$, so that $R_{\mathbb{I}} := \mathbb{I}$ and $u_{\mathbb{I}} := 0$, $v_{\mathbb{I}} := 0$ and lastly $s_{\mathbb{I}} := 0$. As a consequence of $\mathbb{R}^n$ vector spaces forming a group under vector addition and $\mathrm{O}(n)$ itself being a group under multiplication it follows that our maps F defined by the triplet (f, v, s) is thus itself a group having satisfied all the definitions above.

Solution to 9:

We want to find the action of our previous quadruple, instead now on the phase space X. We thus define the following reasonable action and then show it can be made self-consistent. The definition of the action is as follows, given any element $(q,p) \in X$:

$$(u, s, v, R)(q,p) := (Rq + u + \frac{s(Rp + mv)}{m}, Rp + mv),$$

with the defining successive application of two such maps g and g' as $(gg')(q,p) := g''(q,p)$ where the quadruple $g'' = (u'', s'', v'', R'')$ is given in terms of g and g' as follows:

$$(u'', s'', v''R'') := (u + Ru' - s'v, s' + s, v + Rv',\ RR').$$

Solution to 10:

This is self-consistent at least when the choices above are made since (below we use brackets to separate the associative actions):

$$
\begin{aligned}
[gg'](q,p) &= (u+Ru'-s'v, s+s', v+Rv', RR')(q,p) \\
&= (RR'q+u+Ru'-s'v+\frac{(s+s')(RR'p+mv+mRv')}{m}, RR'p+mv+mRv') && (12)\\
&= (RR'q+u+Ru'+\frac{sRR'p}{m}+sv+sRv'+\frac{s'RR'p}{m}+s'Rv', RR'p+mv+mRv') && (13)
\end{aligned}
$$

But on the other hand we have,

$$
\begin{aligned}
g[g'(q,p)] &= (u,s,v,R)[(u',s',v',R')(q,p)] \\
&= (u,s,v,R)(R'q+u'+\frac{s'(R'p+mv')}{m}, R'p+mv') \\
&= (R(R'q+u'+\frac{s'(R'p+mv')}{m})+u+\frac{s(R(R'p+mv')+mv)}{m}, R(R'p+mv')+mv) \\
&= (RR'q+Ru'+\frac{s'RR'p}{m}+s'Rv'+u+\frac{sRR'p}{m}+sRv'+sv, RR'p+mRv'+mv). && (14)
\end{aligned}
$$

Thus we have the associativity of the actions $g(g'(q,p)) = (gg')(q,p)$ since (13) and (14) are identitcal.