Classical Mechanics Homework
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John Baez homework by C.Pro

The Euclidean Group

Recall that the **orthogonal group** $O(n)$ is the group of linear transformations $R: \mathbb{R}^n \to \mathbb{R}^n$ that preserve distances: $||Rx|| = ||x||$ for all $x \in \mathbb{R}^n$. In other words, $O(n)$ is the group of $n \times n$ matrices with $RR^* = 1$.

Define an element of the **Euclidean group** $E(n)$ to be a pair (R, u), where $R \in O(n)$ and $u \in \mathbb{R}^n$. Any element (R, u) gives a transformation of n-dimensional Euclidean space built from an orthogonal transformation and a translation:

$$f_{(R,u)}: \mathbb{R}^n \to \mathbb{R}^n$$

defined by

$$f_{(R,u)})(x) = Rx + u.$$

The map $f_{(R,u)}$ uniquely determines R and u, so we can also think of $E(n)$ as a set of maps.

1. Given two elements $(R, u), (R', u') \in E(n)$, viewed as maps, we have

$$\begin{aligned} f_{(R,u)} \circ f_{(R',u')}(x) &= f_{(R,u)}(R'x + u') \\ &= RR'x + Ru' + u \\ &= f_{(RR',Ru'+u)}(x) \end{aligned}$$

and since $RR' \in O(n)$ and $Ru' + u \in \mathbb{R}^n$, $E(n)$ as maps, is closed under composition.

2. Note that $(1, 0) \in E(n)$ where 1 is the $n \times n$ identity matrix and 0 is the origin in $\mathbb{R}^n$. By the binary operation on $E(n)$ defined above, it's clear that $(1, 0)$ acts as an identity element in this set. Given an element $(R, u) \in E(n)$, note that

$$f_{(R,u)} \circ f_{(R^*,-R^*u)} = f_{(RR^*,R(-R^*u)+u)} = f_{(1,0)}$$

and

$$f_{(R^*,-R^*u)} \circ f_{(R,u)} = f_{(R^*R,R^*u-R^*u)} = f_{(1,0)}$$

so that $(R^*, -R^*u) \in E(n)$ is the inverse to (R, u).

3. Let $(R, u), (R', u'), (R'', u'') \in E(n)$. Since $O(n)$ is associative under composition of linear transformations,

$$((RR')R'', (RR')u'' + (Ru' + u)) = (R(R'R''), R(R'u'' + u') + u),$$

or equivalently,

$$(f_{(R,u)} \circ f_{(R',u')}) \circ f_{(R'',u'')} = f_{(R,u)} \circ (f_{(R',u')} \circ f_{(R'',u'')}),$$

we have $(E(n), \circ)$ is associative as well and thus is a group.

Note that as a set we have $E(n) = O(n) \times \mathbb{R}^n$. However, as a group $E(n)$ is not the direct product of the groups $O(n)$ and $\mathbb{R}^n$, becuase the formulas for multiplication and inverse are not just

$$(R, u)(R', u') = (RR', u + u'), \qquad (R, u)^{-1} = (R^{-1}, -u).$$

Instead, the formulas involve the action of $O(n)$ or $\mathbb{R}^n$, so we say $E(n)$ is a 'semidirect' product of $O(n)$ and $\mathbb{R}^n$.

4. We can more elegantly define the Euclidean group to be the group of all distance-preserving transformations of Euclidean space. Indeed, if $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is a map that preserves distances, that is,

$$||f(x) - f(y)|| = ||x - y||$$

for all $x, y \in \mathbb{R}^n$, then

$$f(x) = Rx + u$$

for some $(R, u) \in E(n)$.

Proof: We begin with the assumption that $f(0) = 0$ and thus for any $x \in \mathbb{R}^n$, $||f(x)|| = ||x||$. Therefore, by showing f is linear, we'll have shown that $f \in O(n)$. Take any $x \in \mathbb{R}^n$ different from zero and any $\alpha \in \mathbb{R}$. We have

$$\begin{aligned}
||\alpha x|| &= ||f(\alpha x)|| \\
&= ||f(\alpha x) - f(x) + f(x)|| \\
&\leq ||f(\alpha x) - f(x)|| + ||f(x)|| \\
&= ||\alpha x - x|| + ||x|| \\
&= ||\alpha x||,
\end{aligned}$$

and thus equality must occur throughout. However, the triangle inequality says that equality holds if and only if the vectors $f(\alpha x) - f(x)$ and $f(x)$ are linearly dependent. Therefore, $f(\alpha x) = \beta f(x)$ and thus $|\alpha| = |\beta|$. In addition,

$$\begin{aligned}
|\alpha - 1|||x|| &= ||\alpha x - x|| \\
&= ||f(\alpha x) - f(x)|| \\
&= ||\beta f(x) - f(x)|| \\
&= |\beta - 1|||f(x)|| \\
&= |\beta - 1|||x||,
\end{aligned}$$

from which we can conclude $|\alpha - 1| = |\beta - 1|$ and these together can only happen if $\alpha = \beta$. Thus,

$$f(\alpha x) = \alpha f(x).$$

Now take any $x, y \in \mathbb{R}^n$ such that $x \neq y$. Let

$$z = \frac{1}{2}f(x + y) - f(x) \qquad \text{and} \qquad w = f(y) - \frac{1}{2}f(x + y)$$

and note that

$$\begin{aligned}
||z|| &= \left|\left|f\left(\frac{x}{2} + \frac{y}{2}\right) - f(x)\right|\right| \\
&= \left|\left|\frac{y}{2} - \frac{x}{2}\right|\right| \\
&= \left|\left|f(y) - f\left(\frac{x}{2} + \frac{y}{2}\right)\right|\right| \\
&= ||w||.
\end{aligned}$$

Note further that

$$\begin{aligned}
||y - x|| &= ||f(y) - f(x)|| \\
&= ||z + w|| \\
&\leq ||z|| + ||w|| \\
&= \frac{1}{2}||y - x|| + \frac{1}{2}||y - x|| \\
&= ||y - x||
\end{aligned}$$

and thus equality occurs throughout. As before, we must have $z = \lambda w$ for some $\lambda \in \mathbb{R}$ and since $||z|| = ||w||$, $\lambda = \pm 1$. If $z = -w$ we have $f(x) = f(y)$ which implies $||x - y|| = ||f(x) - f(y)|| = 0$ or that $x = y$ contrary to the choice of x and y. Therefore $z = w$ and this says

$$f(x + y) = f(x) + f(y),$$

so f is indeed linear and hence $f \in O(n)$.

For the general case, set $f(0) = u$ and define $R(x) = f(x) - u$. Then R is also a map that preserves distances and $R(0) = 0$. From the above, $R \in O(n)$ and thus $f(x) = Rx + u$ as desired.

The Galilei Group

Define an element of the **Galilei group** $G(n+1)$ to be a triple (f, v, s) where $f \in E(n), v \in \mathbb{R}^n$ and $s \in \mathbb{R}$. We call f a **Euclidean transformation**, v a **Galilei boost** and s a **time translation**.

Any element $(f, v, s) \in G(n+1)$ gives a transformation of $(n+1)$-dimensional spacetime

$$F_{(f,v,s)} \colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$$

defined by

$$F_{(f,v,s)}(x, t) = (f(x) + vt, t + s)$$

for all $(x, t) \in \mathbb{R}^{n+1}$. The map $F_{(f,v,s)}$ uniquely determines f, v and s, so we can also think of $G(n+1)$ as a set of maps.

5. Given any $(f_{(R,u)}, v, s), (f_{R',u'}), v', s') \in G(n+1)$, viewed as maps, we have

$$\begin{aligned}
F_{(f_{(R,u)},v,s)} \circ F_{(f_{(R',u')},v',s')}(x,t) &= F_{(f_{(R,u)},v,s)}(R'x + u' + v't, t + s') \\
&= (R(R'x + u' + v't) + u + v(t + s'), (t + s') + s) \\
&= (RR'x + (Ru' + u + vs') + (Rv' + v)t, t + (s' + s)) \\
&= F_{(f_{(RR',Ru'+u+vs')},Rv'+v,s'+s)}(x,t)
\end{aligned}$$

and since $(RR', Ru' + u + vs') \in E(n), Rv' + v \in \mathbb{R}^n$ and $s' + s \in \mathbb{R}$, $G(n+1)$ as maps, is closed under composition. For convenience of notation, we'll henceforth view elements of $G(n+1)$ as 4-tuples (R, u, v, s) where $R \in O(n), u, v \in \mathbb{R}^n$ and $s \in \mathbb{R}$ and take a binary operation on $G(n+1)$ defined above, that is

$$(R, u, v, s)(R', u', v', s') = (RR', Ru' + u + vs', Rv' + v, s' + s).$$

6. Note that $(1_{E(n)}, 0, 0) = (1_{O(n)}, 0, 0, 0) \in G(n+1)$. With this binary operation on $G(n+1)$, it's clear that $(1_{E(n)}, 0, 0)$ acts as an identity element. Now take any $(R, u, v, s) \in G(n+1)$ and note $(R^*, R^*(sv - u), -R^*v, -s)$ is also a member of this set. We have

$$\begin{aligned}
(R, u, v, s)(R^*, R^*(sv - u), -R^*v, -s) &= (RR^*, R(R^*(sv - u)) + u - sv, R(-R^*v) + v, s - s) \\
&= (1_{O(n)}, 0, 0, 0)
\end{aligned}$$

and

$$\begin{aligned}
(R^*, R^*(sv - u), -R^*v, -s)(R, u, v, s) &= (R^*R, R^*u + R^*(sv - u) + -sR^*v, R^*v - R^*v, -s + s) \\
&= (1_{O(n)}, 0, 0, 0).
\end{aligned}$$

Thus, $(R^*, R^*(sv - u), -R^*v, -s)$ acts as the inverse to (R, u, v, s).

7. Associativity is routine to check, but the details will be omitted. Therefore, $G(n+!)$ is indeed a group.

As a set we have $G(n+1) = E(n) \times \mathbb{R}^n \times \mathbb{R}$. However, it is again not the direct product of these groups, but only a semidirect product

8. Consider the trivial vector bundle $T: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ with total space viewed as the real vector space $\mathbb{R}^{n+1}$, base space the time axis $\mathbb{R}$ and fibers as n-dimensional Euclidean space $\mathbb{R}^n$. Denote the fibers over t as $\mathbb{R}^n_t$. Let $S: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ be the projection onto the space axis. Note both S and T are linear maps. If two elements $a, b \in \mathbb{R}^{n+1}$ belong to the same fiber over t, that is $T(b-a) = 0$, we have the concept of distance $|| \cdot ||_t$ between them defined by

$$||b-a||_t = ||S(b-a)||,$$

where the subscript t is used to emphasize that $a, b \in \mathbb{R}^n_t$. However, if a and b are elements of different fibers, we do not have any geometric tools to study them. We turn to the fact spacetime is a real vector space and make the following observation: For every $a, b \in \mathbb{R}^{n+1}$ such that $S(b-a) = 0$, we have $a = (x, t_0)$ and $b = (x, t_1)$ and so

$$b - a = (0, t_1 - t_0) = T(b-a)c.$$

where $c = (0, 1) \in \mathbb{R}^n \times \mathbb{R}$. In words, the vector between the same point at different times is uniquely determined by the amount of time passed.

The above has defined a structure on spacetime as a 4-tuple $(R^n \times \mathbb{R}, p, S, || \cdot ||_t)$

If $F: \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ preserves this structure, that is, for every $a, b \in \mathbb{R}^{n+1}$ we have

$$\begin{aligned} T(b-a) &= T(F(b) - F(a)) && (1) \\ ||b-a||_t &= ||F(b) - F(a)||_{t'} && (2) \end{aligned}$$

where $t' = T(F(a)) = T(F(b))$ and if $S(F(b) - F(a)) = 0$,

$$F(b) - F(a) = T(F(b) - F(a))d \qquad (3)$$

for some $d \in \mathbb{R}^{n+1}$, then $F \in G(n+1)$ and conversely.

Proof: Write $F = (F_\sigma, F_\tau)$ where $F_\sigma: \mathbb{R}^{n+1} \to \mathbb{R}^n$ and $F_\tau: \mathbb{R}^{n+1} \to \mathbb{R}$. Condition (1) is then equivalent to

$$F_\tau(x, t) - F_\tau(x', t') = t - t'$$

for all $(x, t), (x', t') \in \mathbb{R}^{n+1}$. Therefore all partial derivatives of F_τ exist and vanish in all but the time component, that is,

$$\nabla F_\tau = (0, 1).$$

For each $t \in \mathbb{R}$, define $g_t: \mathbb{R}^n \to \mathbb{R}^n$ by $g_t(x) = F_\sigma(x, t)$. Then condition (2) says

$$||g_t(x) - g_t(y)|| = ||F(x, t) - F(y, t)||_{t+s} = ||x - y||.$$

By problem 4, we have $g_t \in E(n)$ and thus we know that g_t has a total derivative and in terms of the Jacobian

$$Jg_t = R \in O(n).$$

For each $x \in \mathbb{R}^n$ define $g_x: \mathbb{R} \to \mathbb{R}^{n+1}$ by $g_x(t) = F(x, t)$. Condition (3) says for every t, t'

$$g_x(t) - g_x(t') = F(x, t) - F(x, t') = T(F(x, t) - F(x, t'))d = (t - t')d$$

and so g_x has a time derivative and

$$\frac{dg_x}{dt} = \frac{\partial F}{\partial t} = d.$$

Since,

$$\frac{\partial F_\tau}{\partial t} = 1$$

we have $d = (v, 1)$ where $v \in \mathbb{R}^n$.

Putting this together, we see that the total derivative of F exists and is constant with

$$JF = \begin{pmatrix} R & v \\ 0 & 1 \end{pmatrix}.$$

Therefore,

$$F(x,t) = \begin{pmatrix} R & v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} + \begin{pmatrix} u \\ s \end{pmatrix}$$

for some $(u, s) \in \mathbb{R}^{n+1}$ or $F(x,t) = (f_{(R,u)}(x) + vt, t + s) \in G(n+1)$ as desired. The converse is immediate.

The Free Particle

We have just described how the Euclidean group acts on Euclidean space and how the Galilei group acts on Galilean spacetime. Now we will figure out how the Galilei group acts on the phase space of a free particle! Recall that the phase space of a particle in n-dimensional Euclidean space is $X = \mathbb{R}^n \times \mathbb{R}^n$, where a point $(q, p) \in X$ describes the particle's position and momentum. The various subgroups of the Galilei group act on X as follows:

• The translation group $\mathbb{R}^n$ is a subgroup of $E(n)$ and thus $G(n+1)$ in an obvious way, and it acts on X as follows:

$$u(q,p) = (q+u, p) \qquad u \in \mathbb{R}^n.$$

In other words, to translate a particle we translate its position but leave its momentum alone!

• The orthogonal group $O(n)$ is also a subgroup of $E(n)$ and thus $G(n+1)$ in an obvious way, and it acts on X as follows:

$$R(q,p) = (Rq, Rp) \qquad R \in O(n).$$

In other words, to rotate a particle we rotate both its position and momentum!

• The group of Galilei boosts $\mathbb{R}^n$ is a subgroup of $G(n+1)$ in an obvious way, and it acts on X as follows:

$$v(q,p) = (q, p - mv) \qquad v \in \mathbb{R}^n.$$

In other words, to boost a particle's velocity by v we subtract mv from its momentum but leave its position alone!

• Finally, the time translation group $\mathbb{R}$ is a subgroup of $G(n+1)$ in an obvious way, and it acts on X as follows:

$$s(q,p) = (q + sp/m, p) \qquad s \in \mathbb{R}.$$

This is where we are assuming the particle is **free**: the force on it is zero, so it moves along at a constant velocity, namely p/m.

9. Consider the map defined by

$$(R, u, v, s)(q,p) = (R(q + sp/m) + (u - sv), Rp - mv).$$

10. The above defines an action of $G(n+1)$ on X

Proof: First recall that $(1, 0, 0, 0) \in G(n+1)$ is the identity element. We have

$$(1,0,0,0)(q,p) = (1q, 1p) = (q,p)$$

for all $(q, p) \in X$ as required. Now take any $g_1 = (R, u, v, s), g_2 = (R', u', v', s') \in G(n+1)$ and note

$$\begin{aligned}
g_1(g_2(q, p)) &= (R, u, v, s)(R'(q + s'p/m) + (u' - s'v'), R'p - mv') \\
&= (R[R'(q + s'p/m) + (u' - s'v') + s(R'p - mv)/m] + (u - sv), R(R'p - mv') - mv) \\
&= (RR'(q + (s' + s)p/m) + ((Ru' + u + vs') - (s' + s)(Rv' + v)), RR'p - m(Rv' - v)) \\
&= (RR', Ru' + u + vs', Rv' + v, s' + s)(q, p) \\
&= ((R, u, v, s)(R', u', v', s'))(q, p) \\
&= (g_1 g_2)(q, p)
\end{aligned}$$

for all $(q, p) \in X$ which completes the proof.