The Kepler Problem Revisited: The Runge–Lenz Vector

Answers to Math 241 Homework
Toby Bartels

1.
$$\begin{aligned}\dot{\mathbf{J}} &= m\dot{\mathbf{q}} \times \dot{\mathbf{q}} + m\mathbf{q} \times \ddot{\mathbf{q}} \\ &= \mathbf{0} - k\mathbf{q} \times \hat{\mathbf{q}}/q^2 \\ &= \mathbf{0}\end{aligned}$$

since cross products of parallel vectors are zero.

2.
$$\begin{aligned}m\ddot{\mathbf{q}} \times \mathbf{J} &= -k\hat{\mathbf{q}} \times \mathbf{J}/q^2 \\ &= -mk\hat{\mathbf{q}} \times (\mathbf{q} \times \dot{\mathbf{q}})/q^2 \\ &= -mk(\hat{\mathbf{q}} \cdot \dot{\mathbf{q}})\mathbf{q}/q^2 + mk(\hat{\mathbf{q}} \cdot \mathbf{q})\dot{\mathbf{q}}/q^2 \\ &= (mk/q^3)\big((\mathbf{q} \cdot \mathbf{q})\dot{\mathbf{q}} - (\mathbf{q} \cdot \dot{\mathbf{q}})\mathbf{q}\big)\end{aligned}$$

so $\ddot{\mathbf{q}} \times \mathbf{J} = (k/q^3)\big((\mathbf{q} \cdot \mathbf{q})\dot{\mathbf{q}} - (\mathbf{q} \cdot \dot{\mathbf{q}})\mathbf{q}\big)$ Now, differentiate $q^2 = \mathbf{q} \cdot \mathbf{q}$ to get $2q\dot{q} = 2\mathbf{q} \cdot \dot{\mathbf{q}}$, and then differentiate $\mathbf{q} = q\hat{\mathbf{q}}$ to get $\dot{\mathbf{q}} = \dot{q}\hat{\mathbf{q}} + q\dot{\hat{\mathbf{q}}}$. Then $q^3\dot{\hat{\mathbf{q}}} = q^2\dot{\mathbf{q}} - q\dot{q}\mathbf{q} = (\mathbf{q} \cdot \mathbf{q})\dot{\mathbf{q}} - (\mathbf{q} \cdot \dot{\mathbf{q}})\mathbf{q}$, and we see that $\ddot{\mathbf{q}} \times \mathbf{J} = k\dot{\hat{\mathbf{q}}}$.

3. Since $\dot{\mathbf{J}} = \mathbf{0}$, we obtain $\frac{d}{dt}(\dot{\mathbf{q}} \times \mathbf{J}) = \ddot{\mathbf{q}} \times \mathbf{J} = k\dot{\hat{\mathbf{q}}}$, as desired.

4. Since k is a constant, the above formula says that

$$\frac{d}{dt}(\dot{\mathbf{q}} \times \mathbf{J} - k\hat{\mathbf{q}}) = \mathbf{0}.$$

Thus if $\mathbf{x} := \dot{\mathbf{q}} \times \mathbf{J} - k\hat{\mathbf{q}}$, then we have $\dot{\mathbf{q}} \times \mathbf{J} = k\hat{\mathbf{q}} + \mathbf{x}$ with $\dot{\mathbf{x}} = \mathbf{0}$.

5. If $\mathbf{A} = \frac{1}{k}\dot{\mathbf{q}} \times \mathbf{J} - \hat{\mathbf{q}}$, then

$$\begin{aligned}\mathbf{A} \cdot \mathbf{q} &= \tfrac{1}{k}(\dot{\mathbf{q}} \times \mathbf{J}) \cdot \mathbf{q} - \hat{\mathbf{q}} \cdot \mathbf{q} \\ &= \tfrac{1}{k}(\mathbf{q} \times \dot{\mathbf{q}}) \cdot \mathbf{J} - q\hat{\mathbf{q}} \cdot \hat{\mathbf{q}} \\ &= \tfrac{1}{km}\mathbf{J} \cdot \mathbf{J} - q\end{aligned}$$

since $\mathbf{J} = m\mathbf{q} \times \dot{\mathbf{q}}$ and $|\hat{\mathbf{q}}| = 1$.

6. $Aq\cos\theta = \mathbf{A} \cdot \mathbf{q} = \frac{1}{km}\mathbf{J} \cdot \mathbf{J} - q$, so $q(A\cos\theta + 1) = \frac{1}{km}\mathbf{J} \cdot \mathbf{J}$, resulting in the desired formula.