Classical Mechanics Homework
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The Kepler Problem

The goal of this problem is to see why particles moving in an inverse square force law — for example, *gravity!* — move along nice curves like ellipses, parabolas and hyperbolas. This is called the **Kepler problem** since it was Kepler who discovered that the orbits of planets were elliptical, and explaining this was the first major triumph of Newtonian mechanics. However, let's start quite generally by studying an arbitrary central force, and specialize to the $1/r^2$ force law of gravity only when that becomes necessary.

Suppose we have a particle moving in a central force. Its position is a function of time, say $q\colon \mathbb{R} \to \mathbb{R}^3$, satisfying Newton's law:

$$m\ddot{q} = f(|q|)\frac{q}{|q|}$$

Here m is its masses, and the force is described by some smooth function $f\colon (0, \infty) \to \mathbb{R}$. Let's write the force in terms of a potential as follows:

$$f(r) = -\frac{dV}{dr}.$$

Using conservation of angular momentum we can choose coordinates where the particle lies in the xy plane at all times. Thus we may assume the z component of $q(t)$ and In short, we have reduced the problem to a 2-dimensional problem!

Now let's work in polar coordinates: the point q lies in the xy plane so write it in polar coordinates as (r, θ). As usual, let's write time derivatives with dots:

$$\dot{r} = \frac{dr}{dt}, \qquad \dot{\theta} = \frac{d\theta}{dt}.$$

Now here's where you come in!

1. Show that the energy E of the particle is given by

$$E = \frac{1}{2}m(r^2\dot{\theta}^2 + \dot{r}^2) + V(r) \tag{1}$$

and the angular momentum J is a vector with vanishing x and y components, and z component given by

$$j = mr^2\dot{\theta}. \tag{2}$$

2. We can use equation (2) to solve for $\dot{\theta}$ in terms of r:

$$\dot{\theta} = \frac{j}{mr^2} \tag{3}$$

Use this and equation (1) to express E in terms of r:

$$E = \frac{1}{2}m\dot{r}^2 + V_{\text{eff}}(r)$$

where

$$V_{\text{eff}}(r) = V(r) + \frac{j^2}{2mr^2}.$$

Thus the energy looks just like the energy of a particle of mass m in a potential V_{eff} on the half-line $\{0 < r < \infty\}$. We have reduced the problem to a 1-dimensional problem! V_{eff} is called the **effective potential**. Note that the second term creates the effect of a repulsive force equal to j^2/mr^3, called the **centrifugal force**.

3. Show that

$$\dot{r} = \sqrt{\frac{2}{m}(E - V_{\text{eff}}(r))}. \tag{4}$$

We could solve this differential equation to find r as a function of t, but it's nicer to find r as a function of θ, since this allows us to see the shape of the particles' orbits. In fact it turns out to be easier to first find θ as a function of r and then solve for r in terms of θ — so that's what we'll do.

4. Using equations (3) and (4) show that

$$\frac{d\theta}{dr} = \frac{j/mr^2}{\sqrt{\frac{2}{m}(E - V_{\text{eff}}(r))}}.$$

Conclude that

$$\theta = \theta_0 + \int \frac{(j/mr^2)\, dr}{\sqrt{\frac{2}{m}(E - V_{\text{eff}}(r))}} \tag{5}$$

Now let's specialize to the case of gravity, where $f(r) = -k/r^2$ and thus $V(r) = -k/r$ for some constant k.

5. Sketch a graph of the effective potential $V_{\text{eff}}(r)$ in this case, and say what a particle moving in this potential would do, depending on its energy E.

6. Show using equation (5) that

$$\theta = \theta_0 + \arccos \frac{\frac{j}{mr} - \frac{k}{j}}{\sqrt{\frac{2E}{m} + \frac{k^2}{j^2}}}.$$

This is the only part of this homework where you really need to *sweat.* However, some ways to do it are easier than others, so think a bit before you plunge into an enormous masochistic calculation — if you do it intelligently, you will only need a *medium-sized* masochistic calculation! For example, you may want to derive a general formula for

$$\int \frac{dx}{\sqrt{ax^2 + bx + c}}$$

and then use it to do the integral in equation (5).

7. Reduce the clutter a bit more by defining

$$p = j^2/km, \qquad e = \sqrt{1 + \frac{2Ej^2}{mk^2}}.$$

Show that in terms of these variables we have

$$\theta = \theta_0 + \arccos\left(\frac{p/r - 1}{e}\right)$$

and thus

$$r = \frac{p}{1 + e\cos(\theta - \theta_0)}. \tag{6}$$

Note that when $\theta = \theta_0$ the denominator is maximized, so r is minimized. We call this point the *perihelion* of the orbit, since in Newton's original application to the earth going around this sun, this is the point on the earth's orbit where its distance to the sun is minimized.

8. Show that equation (6) describes an ellipse, parabola or hyperbola in polar coordinates, depending on the value of the parameter e, which we call the **eccentricity**. To do this, first simplify things by rotating the coordinate system so that $\theta_0 = 0$. Then express the variables r, θ in terms of x, y and show that equation (6) becomes the equation

$$(1 - e^2)x^2 + 2epx + y^2 = p^2.$$

Show that for $e = 0$ this describes a circle of radius p. Show also that for $0 < e < 1$ it describes an ellipse, for $e = 1$ it describes a parabola, and for $e > 1$ it describes a hyperbola. Newton used the elliptic case to predict when the comet discovered by Edmund Halley would return! However, he didn't give Halley much credit for obtaining the necessary data.

9. How are the three kinds of orbits — ellipse, parabola or hyperbola — related to the energy E?