The Kepler Problem

(Background) Suppose we have a particle moving in a central force. Its position is a function of time, say \( q : \mathbb{R} \to \mathbb{R}^3 \), satisfying Newton’s law:

\[
m \ddot{q} = f(|q|) \frac{q}{|q|}
\]

Here \( m \) is its masses, and the force is described by some smooth function \( f : (0, \infty) \to \mathbb{R} \). Let’s write the force in terms of a potential as follows:

\[
f(r) = -\frac{dV}{dr}.
\]

Using conservation of angular momentum we can choose coordinates where the particle lies in the \( xy \) plane at all times. Thus we may assume the \( z \) component of \( q(t) \) and \( \dot{q}(t) \) vanish for all \( t \). In short, we have reduced the problem to a 2-dimensional problem!

Now let’s work in polar coordinates: the point \( q \) lies in the \( xy \) plane so write it in polar coordinates as \((r, \theta)\). As usual, let’s write time derivatives with dots:

\[
\dot{r} = \frac{dr}{dt}, \quad \dot{\theta} = \frac{d\theta}{dt}.
\]

1. Show that the energy \( E \) of the particle is given by

\[
E = \frac{1}{2} m (r^2 \dot{\theta}^2 + \dot{r}^2) + V(r)
\]

and the angular momentum \( J \) is a vector with vanishing \( x \) and \( y \) components, and \( z \) component given by

\[
j = mr^2 \dot{\theta}.
\]

Recall that the energy of such a particle is given by

\[
E = \frac{1}{2} m \dot{q}(t)^2 + V(|q(t)|).
\]

Noting that in polar coordinates

\[
\dot{q} = (\dot{r} \cos \theta - r \dot{\theta} \sin \theta, \dot{r} \sin \theta + r \dot{\theta} \cos \theta),
\]

we see that

\[
\dot{q}(t)^2 = \dot{r}^2 \cos^2 \theta - r \dot{r} \dot{\theta} \sin 2\theta + r^2 \dot{\theta}^2 \sin^2 \theta + \dot{r}^2 \sin^2 \theta + r \dot{r} \dot{\theta} \sin 2\theta + r^2 \dot{\theta}^2 \cos^2 \theta
\]

which reduces nicely to \( \dot{q}(t)^2 = \dot{r}^2 + r^2 \dot{\theta}^2 \). Substitution of this last expression for \( \dot{q}(t)^2 \) into (3) and noting that \( r = |q(t)| \) yields (1).

Now we will show that the angular momentum \( J \) is a vector with vanishing \( x \) and \( y \) components with the \( z \) component given by (2). The angular momentum is
\[ J = mq \times \dot{q} \]

and if we use the expression for \( \dot{q} \) obtained in (4), we have

\[ q \times \dot{q} = (r \cos \theta i + r \sin \theta j) \times [(\dot{r} \cos \theta - r \dot{\theta} \sin \theta)i + (\dot{r} \sin \theta + r \dot{\theta} \cos \theta)j] \]
\[ = [r^2 \dot{\theta} \cos^2 \theta + \dot{r} \cos \theta \sin \theta - r \sin \theta (\dot{r} \cos \theta - r \dot{\theta} \sin \theta)]k \]
\[ = r^2 \dot{\theta} k, \]

so that \( J = mr^2 \dot{\theta} k \).

2. We use equation (2) to solve for \( \dot{\theta} \) in terms of \( r \):

\[ \dot{\theta} = \frac{j}{mr^2} \]  \hspace{1cm} (5)

Combining this and equation (1) we express \( E \) in terms of \( r \):

\[ E = \frac{1}{2}mr^2 + V_{\text{eff}}(r) \]  \hspace{1cm} (6)

where

\[ V_{\text{eff}}(r) = V(r) + \frac{j^2}{2mr^2}. \]

The only thing to note here is that \( \dot{\theta}^2 = j^2/m^2r^4 \).

3. We solve (6) for \( \dot{r} \) to obtain

\[ \dot{r} = \sqrt{\frac{2}{m}(E - V_{\text{eff}}(r))}. \]  \hspace{1cm} (7)

It should be noted that in our use of the symbol for the positive square root we are not asserting that \( \dot{r} \) is positive! It is entirely possible that the above root is negative! This, as we will discuss below (in # 5) will not effect the form of our solution for \( r \) in terms of \( \theta \).

4. Using (5) and (7) show that

\[ \frac{d\theta}{dr} = \frac{\dot{\theta}}{\sqrt{\frac{2}{m}(E - V_{\text{eff}}(r))}} = \frac{j/mr^2}{\sqrt{\frac{2}{m}(E - V_{\text{eff}}(r))}}. \]

By the chain rule, we have that

\[ \dot{\theta} = \frac{d\theta}{dr} \dot{r}, \]

which when combined with (7) (and subsequently (5)) gives:

\[ \frac{d\theta}{dr} = \frac{\dot{\theta}}{\sqrt{\frac{2}{m}(E - V_{\text{eff}}(r))}} = \frac{j/mr^2}{\sqrt{\frac{2}{m}(E - V_{\text{eff}}(r))}}. \]

Upon integration we arrive at

\[ \theta = \theta_0 + \int \frac{(j/mr^2)dr}{\sqrt{\frac{2}{m}(E - V_{\text{eff}}(r))}}. \]  \hspace{1cm} (8)
Now let’s specialize to the case of gravity, where \( f(r) = -\frac{k}{r^2} \) and thus \( V(r) = -\frac{k}{r} \) for some constant \( k \).

5. Sketch a graph of the effective potential \( V_{\text{eff}}(r) \) in this case, and say what a particle moving in this potential would do, depending on its energy \( E \).

![Graphs of effective potential](image)

Figure I shows a sketch of \( V_{\text{eff}} \) in the case that \( |j| > m \) (this is the case where \( V_{\text{eff}}'(r) < 0 \) for \( r < j^2/(2mk) \)) and II shows \( V_{\text{eff}} \) where \( |j| < m \) (where \( V_{\text{eff}}'(r) > 0 \) for \( r < j^2/(2mk) \)). In both sketches, the zero is at \( r = j^2/(2mk) \) and the \( r \)-axis is a horizontal asymptote as \( r \to \infty \).

Let us briefly discuss the behavior of a particle with energy \( E < 0 \) with \( |j| > m \). Such a particle is shown in III. As was discussed in the example in class, the particles radius \( r \) would oscillate within the classically allowed region (the \( r \) values lying between the intersection points of \( E \) and \( V_{\text{eff}}(r) \)). The particle would be moving fastest at the minimum value of \( V_{\text{eff}} \) and would change from moving away from the origin to moving towards it (or vice a versa) at the intersection points.

6. Carry out the integration in (8).

We must compute

\[
\int \frac{(j/mr^2)dr}{\sqrt{\frac{2}{m}(E + k/r - j^2/(2mr^2))}}
\]
Too much has been made of this bugaboo! Let’s put this “beast” to rest by an elementary trigonometric substitution:

\[
\frac{j}{m} \left( \frac{1}{r} - \frac{mk}{j^2} \right) = \sqrt{\frac{2E}{m} + \frac{k^2}{j^2}} \cos u.
\]

(The sign of the radical here is chosen to match the sign of the radical in # 3) This substitution comes from completing the square under the radical—a simple and computationally economical process—and recalling the pythagorean identity for sine and cosine. All showboating aside, we see that

\[
(j/mr^2)dr = \sin u \sqrt{\frac{2E}{m} + \frac{k^2}{j^2}} du,
\]

and upon substitution, the integral becomes

\[
\int u \, du = u
\]

(the constant of integration already being accounted for in \(\theta_0\), and any sign changes from radicals canceling). Reversing the trigonometric substitution we see that \(u\) and hence the sought after antiderivative is

\[
\arccos \left( \frac{j}{mr^2} - \frac{k}{j} \right) \sqrt{\frac{2E}{m} + \frac{k^2}{j^2}}.
\]

Whence,

\[
\theta = \theta_0 + \arccos \left( \frac{j}{mr^2} - \frac{k}{j} \right) \sqrt{\frac{2E}{m} + \frac{k^2}{j^2}}. \tag{9}
\]

7. Reduce the clutter in (9) by defining

\[
p = j^2/km, \quad e = \sqrt{1 + \frac{2Ej^2}{mk^2}}.
\]

Note that

\[
\sqrt{\frac{2E}{m} + \frac{k^2}{j^2}} = \frac{k}{j} e,
\]

so that

\[
\frac{j}{mr^2} - \frac{k}{j} = \frac{j}{k} \frac{j}{e} - \frac{k}{j} = \frac{p/r - 1}{e},
\]

from whence it follows that

\[
\theta = \theta_0 + \arccos \left( \frac{p/r - 1}{e} \right).
\]

Solving for \(r\) yields:

\[
r = \frac{p}{1 + e \cos(\theta - \theta_0)}. \tag{10}
\]
We should note that if the sign of the radical for $\dot{r}$

8. Show that equation (10) describes an ellipse, parabola, or hyperbola in polar coordinates, depending on the value of the parameter $e$, which we call the **eccentricity**.

Begin by making a shift (a rotation) of $\theta_0$ in $\theta$. We will call the new coordinates that result from this shift $r'$ and $\theta'$. We have that

$$r' = \frac{p}{1 + e \cos \theta'}$$

by (10), or equivalently

$$r' + er' \cos \theta' = p.$$  

Making the standard change to cartesian coordinates, the above reads

$$\sqrt{x^2 + y^2} + ex = p.$$  

Now a little algebra yields

$$x^2 + y^2 = p^2 - 2ex + e^2 x^2,$$

or put a little differently,

$$(1 - e^2)x^2 + 2ex + y^2 = p^2; \quad (11)$$

which we immediately recognize as the equation of a conic.

The particular conic that (11) describes will be determined by the value of $e$. If $e = 0$, for instance, then (11) reduces to

$$x^2 + y^2 = p^2,$$

a circle centered at the origin with radius $p$. If $e = 1$, then (11) reduces to

$$2(x - p^2/2) = y^2,$$

a parabola with vertex (in the original polar coordinates) $(p^2/2, \theta_0)$ opening **towards** the origin.

Let’s exhaust all of the cases. If $e \neq 0$ or 1, then we may rewrite (11) as

$$\left( \frac{x + \frac{ep}{1-e^2}}{p^2 \frac{1+e^2}{1-e^2}} \right)^2 + \frac{y^2}{p^2 (1+e^2)} = 1. \quad (12)$$

We see that in this case (12) represents either a hyperbola ($e > 1$) with vertices (in rotated cartesian coordinates)

$$\left( \frac{-ep}{1-e^2}, \pm p(1-e^2)^{1/2} \right)$$

opening in the $y$ direction or an ellipse ($0 < e < 1$) with center (again in rotated cartesian coordinates)

$$\left( \frac{-ep}{1-e^2}, 0 \right).$$

9. **How are the three kinds of orbits related to the energy $E$?**

   Recall that $e$ is given by
\[ e = \sqrt{1 + \frac{2Ej^2}{mk^2}}, \]

so that

\[ E = \frac{mk^2}{2j^2}(e^2 - 1). \]

Using this, we compile the following chart:

<table>
<thead>
<tr>
<th>Orbit (type)</th>
<th>Energy</th>
</tr>
</thead>
<tbody>
<tr>
<td>Circular</td>
<td>( E = -\frac{mk^2}{2j^2} )</td>
</tr>
<tr>
<td>Parabolic</td>
<td>( E = 0 )</td>
</tr>
<tr>
<td>Hyperbolic</td>
<td>( E &gt; 0 )</td>
</tr>
<tr>
<td>Elliptic</td>
<td>( -\frac{mk^2}{2j^2} &lt; E &lt; 0 )</td>
</tr>
</tbody>
</table>