Classical Mechanics Homework

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The Kepler Problem.

Suppose we have a particle moving in a central force. Its position is a function of time, say $q\colon \mathbb{R} \to \mathbb{R}^3$, satisfying Newton's law:

$$m\ddot{q} = f(|q|)\frac{q}{|q|}$$

Here m is its masses, and the force is described by some smooth function $f\colon (0,\infty) \to \mathbb{R}$. Let's write the force in terms of a potential as follows:

$$f(r) = -\frac{dV}{dr}.$$

Using conservation of angular momentum we can choose coordinates where the particle lies in the xy plane at all times. Thus we may assume the z component of $q(t)$ and $\dot{q}(t)$ vanish for all t. In short, we have reduced the problem to a 2-dimensional problem!

Now let's work in polar coordinates: the point q lies in the xy plane so write it in polar coordinates as (r, θ). As usual, let's write time derivatives with dots:

$$\dot{r} = \frac{dr}{dt}, \qquad \dot{\theta} = \frac{d\theta}{dt}.$$

1. For the moment, view $\mathbb{R}^2$ as $\mathbb{C}$ and write $q = re^{i\theta}$ where q, r and θ depend on time t. In this guise,

$$\dot{q} = (\dot{r} + i\dot{\theta}r)e^{i\theta}$$

and as a vector dot product,

$$\dot{q}^2 = |\dot{q}|^2 = |\dot{r} + i\dot{\theta}r|^2 = \dot{r}^2 + \dot{\theta}^2 r^2.$$

Therefore, the energy E of the particle is given by

$$\begin{aligned} E &= \frac{m}{2}\dot{q}^2 + V(r) \\ &= \frac{m}{2}(\dot{r}^2 + \dot{\theta}^2 r^2) + V(r). \end{aligned} \tag{1}$$

Let $J = q(t) \times p(t)$, where p is the momentum of q, be the angular momentum of q. Angular momentum in a central force is constant, and thus we are allowed to view q and $p = m\dot{q}$ as points in $\mathbb{C}$. As a vector cross product, J is therefore directed perpendicular to $\mathbb{C}$ so has z component equal to its magnitude

$$\begin{aligned} j &= \mathrm{Im}\,(\bar{q}p) \\ &= \mathrm{Im}\,(\bar{q}m\dot{q}) \\ &= \mathrm{Im}\,(rm\dot{r} + imr^2\dot{\theta}) \\ &= mr^2\dot{\theta}. \end{aligned} \tag{2}$$

2. Using equation (2), we have

$$\dot{\theta} = \frac{j}{mr^2} \tag{3}$$

so that by (1)

$$\begin{aligned} E &= \frac{m}{2}(\dot{r}^2 + \left(\frac{j}{mr^2}\right)^2 r^2) + V(r). \\ &= \frac{m}{2}\dot{r}^2 + \frac{j^2}{2mr^2} + V(r) \\ &= \frac{m}{2}\dot{r}^2 + V_{\text{eff}}(r) \end{aligned}$$

where

$$V_{\text{eff}} = \frac{j^2}{2mr^2} + V(r).$$

Thus the energy looks just like the energy of a particle of mass m in a potential V_{eff} on the half-line $\{0 < r < \infty\}$. We have reduced the problem to a 1-dimensional problem! V_{eff} is called the **effective potential**. Note that the first term creates the effect of a repulsive force equal to j^2/mr^3, called the **centrifugal force**.

3. From this new description of energy, we have

$$\dot{r} = \pm\sqrt{\frac{2}{m}(E - V_{\text{eff}}(r))} \tag{4}$$

so that from (3)

$$\frac{d\theta}{dr} = \frac{\dot{\theta}}{\dot{r}} = \pm\frac{j^2/mr^2}{\sqrt{\frac{2}{m}(E - V_{\text{eff}}(r))}}.$$

We now have θ as a function of r since

$$\begin{aligned} \theta(r) &= \theta(r_0) + \int_{r_0}^{r} \frac{d\theta}{dr}(s)ds \\ &= \theta(r_0) \pm \int_{r_0}^{r} \frac{j^2/ms^2}{\sqrt{\frac{2}{m}(E - V_{\text{eff}}(s))}}ds \end{aligned} \tag{5}$$

for any r_0 in $(0, \infty)$.

Now let us specialize to the case of gravity, where $f(r) = -k/r^2$ and thus $V(r) = -k/r$ for some constant k.

5. First let's examine the graph of the effective potential:

$$V_{\text{eff}}(r) = \frac{j^2}{2mr^2} - \frac{k}{m}.$$

For a given total energy E of q, we have from equation (4) the classically allowed values for r (i.e., $\{r \mid V_{\text{eff}}(r) \leq E\}$). In this case, we see from the following figure, a set of values for E which will yield a unique behavior for r as a function of t. In addition, equation (4) gives us the analogy between $\dot{r}$ with respect to r and that of the velocity of a skateboarder with respect to her position on the ramp $(r, V_{\text{eff}}(r))$ under a roof of height E.

Suppose the total energy of q is E_1 and is equal to the minimum value of the effective potential. In this case there is only one classically allowed value:

$$r = \frac{j^2}{mk}$$

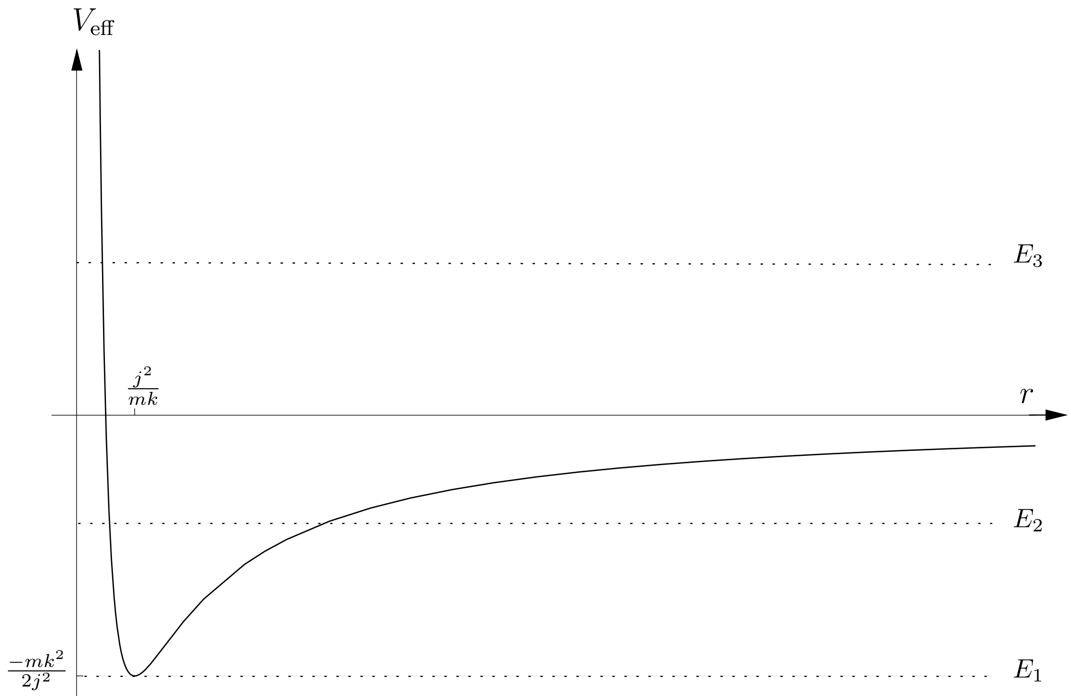

Figure 1: Gravitational effective potential

and the only interesting thing here is to ask about the mood of the skateboarder.

Suppose now that the total energy is E_2 such that

$$\frac{-mk^2}{2j^2} < E_2 < 0.$$

We see the radius accelerate down the ramp away from the attracting body, then decelerate up the ramp, touch the roof and return with the same speed pattern to do it again.

If the total energy E_3 is such that $E_3 \geq 0$ we have enough energy to escape the pull of the attracting body and the distance from this object becomes unbounded. It should be noted, when $E_3 = 0$ we have the minimum value of energy to escape this pull

6. Now, to find θ as a function of r, by equation (5) we must find the following antiderivative:

$$\int \frac{j/r^2}{\sqrt{-j^2(\frac{1}{r^2}) + 2km(\frac{1}{r}) + 2mE}} dr.$$

Use the substitution $x = j/r$ to transform the above into

$$\int \frac{-dx}{\sqrt{ax^2 + bx + c}}$$

where $a = -1$, $b = 2km/j$ and $c = 2mE$.

To solve this problem, first set α to satisfy

$$\cos\alpha = \frac{x + \frac{b}{2a}}{\sqrt{\frac{b^2-4ac}{4a^2}}}. \tag{6}$$

Then by the Pythagorean Theorem,

$$\csc \alpha = \frac{\sqrt{\frac{b^2-4ac}{4a^2}}}{\sqrt{ax^2+bx+c}}.$$

By differentiating (6), then substituting the above, we obtain

$$d\alpha = -\csc \alpha \sqrt{\frac{4a^2}{b^2-4ac}}\, dx = \frac{-dx}{\sqrt{ax^2+bx+c}}.$$

Therefore,

$$\alpha + C = \int d\alpha = \int \frac{-dx}{\sqrt{ax^2+bx+c}}$$

and thus

$$\int \frac{-dx}{\sqrt{ax^2+bx+c}} = C + \arccos \frac{x+\frac{b}{2a}}{\sqrt{\frac{b^2-4ac}{4a^2}}}.$$

Backtracking to θ-ville, from equation (5), we have

$$\theta(r) = \theta_0 \pm \arccos \frac{\frac{j}{mr}-\frac{k}{j}}{\sqrt{\frac{2E}{m}+\frac{k^2}{j^2}}}.$$

7. By rearranging letters in our formula for θ, we get

$$\theta(r) = \theta_0 \pm \arccos \frac{\frac{j^2}{mrk}-1}{\sqrt{1+\frac{2Ej^2}{mk^2}}}.$$

Now if we set

$$p = j^2/km, \qquad e = \sqrt{1+\frac{2Ej^2}{mk^2}},$$

we have

$$\theta(r) = \theta_0 \pm \arccos \left(\frac{p/r-1}{e}\right).$$

This equation can be solved for r in terms of θ as:

$$r(\theta) = \frac{p}{1+e\cos(\theta-\theta_0)}. \tag{7}$$

Note that when $\theta = \theta_0$ the denominator is maximized, so r is minimized. We call this point the *perihelion* of the orbit, since in Newton's original application to the earth going around this sun, this is the point on the earth's orbit where its distance to the sun is minimized.

8. Now set $w(t) = e^{i\theta_0}q(t)$ and say $w(t)$ has coordinates $(x(t), y(t))$. Then equation (7) becomes

$$r + ex = p$$

from which we can square both sides and get

$$(1-e^2)x^2 + 2epx + y^2 = p^2. \tag{8}$$

Now if $e = 0$, then (8) becomes

$$x^2 + y^2 = p^2,$$

thus $w(t)$, and hence $q(t)$ is on the circle of radius p.

If $0 < e < 1$, then $1 - e^2 > 0$ so that (8) can be written as

$$\frac{\left(x + \frac{pe}{1-e^2}\right)^2}{\left(\frac{p}{1-e^2}\right)^2} + \frac{y^2}{\left(\frac{p}{\sqrt{1-e^2}}\right)^2} = 1,$$

thus $w(t)$, and hence $q(t)$ is on an ellipse with foci at $(0, 0)$ and $(\frac{-pe}{1-e^2}, 0)$.

If $e = 1$, solving for x in (8) gives

$$x = -\frac{y^2}{2p} + \frac{p}{2},$$

thus $q(t)$ is on a parabola.

If $e > 1$, then $e^2 - 1 > 0$ so that (8) can be written as

$$\frac{\left(x + \frac{pe}{1-e^2}\right)^2}{\left(\frac{p}{1-e^2}\right)^2} - \frac{y^2}{\left(\frac{p}{\sqrt{e^2-1}}\right)^2} = 1,$$

and hence $q(t)$ is on an hyperbola with foci at $(0, 0)$ and $(\frac{-pe}{1-e^2}, 0)$.

9. So when $e = 0$ we have an orbit of a circle, solving for the total energy we see

$$E = \frac{-mk^2}{2j^2}$$

which is the minimum value of the effective potential as expected. When $0 < e < 1$ we're on an ellipse, and this occurs when $-mk^2/2j^2 < E < 0$. When $E = 0$, $e = 1$ and we're on a parabola. And when $E > 0$, $e > 1$ so we're on a hyperpola.