The Kepler Problem

1. Show that the energy of a particle is given by \( E = \frac{1}{2} m (r^2 \dot{\theta}^2 + \dot{r}^2) + V(r) \) and the angular momentum \( J \) has \( z \) coordinate \( j = mr^2 \dot{\theta} \) and vanishing \( x \) and \( y \) coordinates.

   Solution: We know \( E = T + V \) where \( T \) is the kinetic energy and \( V \) is the potential energy. Kinetic energy is \( T = \frac{1}{2} m (\dot{q}(t) \cdot \dot{q}(t)) \) where \( q(t) = (r(t) \cos \theta(t), r(t) \sin \theta(t), 0) \). Since \( \dot{q}(t) = (\dot{r} \cos \theta - r \dot{\theta} \sin \theta, \dot{r} \sin \theta + r \dot{\theta} \cos \theta, 0) \), \( \dot{q} \cdot \dot{q} = \dot{r}^2 + r^2 \dot{\theta}^2 \) and \( T = \frac{1}{2} m (r^2 \dot{\theta}^2 + \dot{r}^2) \). By assumption, \( V \) depends on \( r \), since the force depends only on \( r \).

   The angular momentum is given \( j = q(t) \times m \dot{q}(t) \). Since \( q \) and \( \dot{q} \) lie in the \( xy \) plane, \( q(t) \times \dot{q}(t) = (0, 0, r^2 \dot{\theta}) \) and so the \( z \) coordinate of the angular momentum is \( mr^2 \dot{\theta} \).

2. Use the angular momentum to solve for \( \dot{\theta} \) and write \( E \) as \( E = \frac{1}{2} mr^2 + V_{\text{eff}}(r) \) where \( V_{\text{eff}}(r) = V(r) + \frac{j^2}{2mr^2} \).

   Solution: Since \( j = mr^2 \dot{\theta} \), we have \( \dot{\theta} = \frac{j}{mr^2} \). The equation for \( T \) gives \( T = \frac{1}{2} m \left( \dot{r}^2 \left( \frac{j}{mr^2} \right)^2 + \dot{r}^2 \right) \) and \( E = \frac{1}{2} mr^2 + \frac{j^2}{2mr^2} + V(r) = \frac{1}{2} mr^2 + V_{\text{eff}}(r) \), where \( V_{\text{eff}}(r) = \frac{j^2}{2mr^2} + V(r) \).

3. Show that \( \dot{r} = \sqrt{\frac{2}{m}(E - V_{\text{eff}}(r))} \).

   Solution: Solving \( E = \frac{1}{2} mr^2 + V_{\text{eff}}(r) \) for \( \dot{r} \) gives \( \dot{r} = \frac{2}{m} (E - V_{\text{eff}}(r)) \). Taking the square root gives \( \dot{r} = \frac{\sqrt{2}}{m} (E - V_{\text{eff}}(r)) \).

4. Show that \( \frac{d\theta}{dr} = \frac{j/mr^2}{\sqrt{\frac{2}{m}(E - V_{\text{eff}}(r))}} \).

   Solution: By the chain rule \( \frac{d\theta}{dr} = \frac{d\theta}{dt} \frac{dt}{dr} \). Implicit differentiation (like in 9A) gives \( \frac{dt}{dr} = \frac{1}{\sqrt{\frac{2}{m}(E - V_{\text{eff}}(r))}} \), provided \( \dot{r} > 0 \). Since \( \frac{d\theta}{dt} = \frac{j}{mr^2} \), we have \( \frac{d\theta}{dr} = \frac{j/mr^2}{\sqrt{\frac{2}{m}(E - V_{\text{eff}}(r))}} \).

   Integrating this gives \( \theta = \theta_0 + \int \frac{(j/mr^2)dr}{\sqrt{\frac{2}{m}(E - V_{\text{eff}}(r))}} \).
5. Sketch the graph of \( V_{\text{eff}}(r) \) when \( V = -\frac{k}{r} \) and describe what a particle in this potential would do, depending on its energy \( E \).

Solution: We have \( V_{\text{eff}}(r) = \frac{j^2}{2mr^2} - \frac{k}{r} = k \left( \frac{j^2}{2mk} - \frac{1}{r^2} \right) \). This is minimized when \( r = \frac{j^2}{mk} \) and then \( V_{\text{eff}} = -\frac{k^2m}{2j^2} \). Since \( \frac{1}{2}mr^2 \) must be nonnegative, we know that a particle’s total energy can never be less than \(-\frac{k^2m}{2j^2}\). If a particle’s total energy is positive, then \( r \) can go to infinity. If \( E < 0 \), then the particle will stay in between two values of \( r \), determined by when the energy equals \( V_{\text{eff}} \).

6. Show \( \theta = \theta_0 + \arccos \frac{\frac{j}{m} - \frac{k}{j}}{\sqrt{\frac{2E}{m} + \frac{k^2}{j^2}}} \).

Solution: \( \int \frac{(j/mr^2)dr}{\sqrt{\frac{2}{m}(E - V_{\text{eff}}(r))}} = \int \frac{(j/mr^2)dr}{\sqrt{\frac{2}{m}(E + \frac{k}{r} - \frac{j^2}{2mr^2})}} = -\int \frac{dw}{\sqrt{-\frac{2Em}{j^2} + \frac{2km}{j}w - w^2}} \), where \( w = \frac{1}{r} \). Using the fact \( \int \frac{dx}{\sqrt{ax^2 + bx + c}} = \frac{1}{\sqrt{-a}} \arccos \left( \frac{-2ax - b}{\sqrt{b^2 - 4ac}} \right) \) when \( a < 0 \), we have

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-\int \frac{dw}{\sqrt{\frac{2Em}{j^2} + \frac{2km}{j}w - w^2}} = \arccos \left( \frac{2w - \frac{2km}{j^2}}{\sqrt{\frac{4k^2m^2}{j} + 42Em}} \right) = \arccos \left( \frac{\frac{j}{m} - \frac{k}{j}}{\sqrt{\frac{2Em}{j^2} + \frac{k^2}{j^2}}} \right). \]

So \( \theta = \theta_0 + \arccos \frac{\frac{j}{m} - \frac{k}{j}}{\sqrt{\frac{2Em}{j^2} + \frac{k^2}{j^2}}} \).
7. Letting $p = \frac{j^2}{km}$ and $e = \sqrt{1 + \frac{2E_j^2}{mk^2}}$, show $\theta = \theta_0 + \arccos\left(\frac{p}{r} - 1\right)$.

Solution: $\theta = \theta_0 + \arccos\frac{j}{\sqrt{\frac{2E_j^2}{mk^2} + 1}} = \theta_0 + \arccos\frac{j}{\sqrt{\frac{2E_j^2}{mk^2} + 1}} = \theta_0 + \arccos\left(\frac{p}{r} - 1\right)$. Solving for $r$ gives $r = \frac{p}{1 + e \cos(\theta - \theta_0)}$.

8. The equation $r = \frac{p}{1 + e \cos(\theta - \theta_0)}$ describes an ellipse, parabola or hyperbola based on the value of $e$.

Solution: By a rotation, we can assume $\theta_0 = 0$. So we have $p = r + er \cos \theta$ or $p = \sqrt{x^2 + y^2 + ex}$. So $x^2 + y^2 = p^2 - 2epx + x^2$, or $(1 - e^2)x^2 + 2epx + y^2 = p^2$.

If $e = 0$, we have $p^2 = x^2 + y^2$, the equation of a circle.

If $0 < e < 1$, let $k = \frac{ep}{1 - e^2}$. Then we have $x^2 + 2kx + \frac{y^2}{1 - e^2} = \frac{p^2}{1 - e^2} = (x - k)^2 + \frac{y^2}{1 - e^2} = \frac{p^2}{1 - e^2} + k^2$, and ellipse.

If $e = 1$, we have $2px = -y^2 + p^2$, a parabola.

If $e > 1$, let $k = \frac{ep}{1 - e^2}$. Then we have $x^2 + 2kx + \frac{y^2}{1 - e^2} = \frac{p^2}{1 - e^2} = (x - k)^2 - \frac{y^2}{e^2 - 1} = \frac{p^2}{1 - e^2} + k^2$, a hyperbola.

9. How are the 3 kinds of orbits related to energy?

Solution: The hyperbola corresponds to $e > 1$, which requires the energy to be positive. The parabola correspond to $e = 1$, which requires energy to be 0. The ellipse corresponds to $0 < e < 1$ which corresponds which requires energy to be negative.