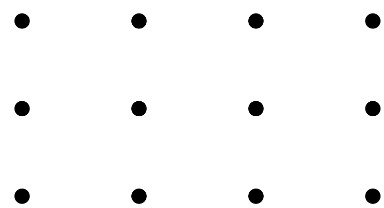
The Pendulum, Elliptic Functions and Imaginary Time

Math 241 Homework
John Baez

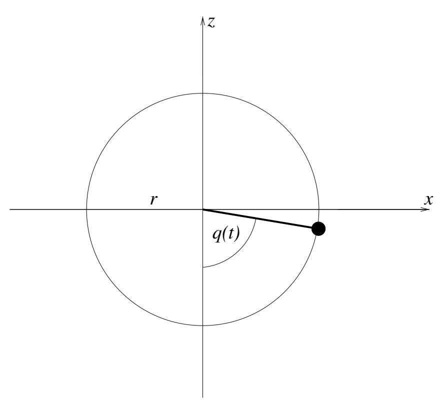
The sine and cosine functions are analytic on the entire complex plane, and also periodic in one direction. It's interesting to look for nice functions that are periodic in *two* directions on the complex plane. Such a function can't be analytic everywhere — it must have poles — since otherwise it would be bounded. Apart from that it can be very nice: it can be analytic except at poles that form a lattice in the complex plane, like this:



A function like this is called an **elliptic function**. Since the plane modulo a lattice is a torus, you can also think of an elliptic function as a function from the torus to the Riemann sphere. The complex plane modulo a lattice is also called an **elliptic curve**; these are important examples of Riemann surfaces.

Jacobi, Weierstrass and other mathematicians did a lot of work on elliptic functions in the 1800s. Elliptic functions and elliptic curves have many applications to number theory, ultimately leading to very deep results such as Wiles' proof of Fermat's theorem. They also have lots of applications to physics — and here you will learn about one of the simplest! If you're feeling less ambitious, do problems 1-11. If you're feeling more ambitious, do problem 12.

Start with a pendulum where a particle of mass m is constrained by a rod to lie on a circle of radius r in the xz plane:



To keep things simple we'll neglect the mass of the rod. The position of the pendulum is a function

$$q\colon \mathbb{R} \to S^1,$$

that is, a function of time taking values in the circle. Concretely we will think of $q(t)$ as the angle counterclockwise from the downwards z axis. However, we are allowed to treat this angle as a real number only if we remember that two angles describe the same position of the pendulum when they differ by an integral multiple of 2π. Thus $q(t)$ is really a member, not of $\mathbb{R}$, but of the quotient group $\mathbb{R}/2\pi\mathbb{Z}$, which is a way of thinking of the circle S^1.

Since the position of the pendulum at time t is really a point $q(t) \in S^1$, it follows that $\dot{q}(t)$ is really an element of the tangent space $T_{q(t)}S^1$. Similarly, the corresponding momentum $p(t)$ lies in the cotangent space $T^*_{q(t)}S^1$, and the state of the pendulum is described by a point $(q(t), p(t))$ in T^*S^1. However, we can treat the time derivative of an angle as a real number using the isomorphism $S^1 \cong \mathbb{R}/2\pi\mathbb{Z}$, so we get an isomorphism $T_{q(t)}S^1 \cong \mathbb{R}$, and thus an isomorphism $T^*_{q(t)}S^1 \cong \mathbb{R}$. We thus have

$$T^*S^1 \cong S^1 \times \mathbb{R}$$

and using this we can think of $(q(t), p(t))$ as a point in $S^1 \times \mathbb{R}$. Of course, most physicists do all this without making such a fuss about it!

Now here's where you come in....

1. Using what you already know about the mechanics of point particles, show that the kinetic energy of the pendulum is

$$K(\dot{q}) = \frac{1}{2}mr^2\dot{q}^2$$

2. Assuming the force of gravity is a vector pointing down with magnitude mG, show that we can assume the potential energy of the pendulum to be

$$V(q) = -mG\cos q.$$

3. Using what you know about classical mechanics on a Riemannian manifold, show that the Hamiltonian of the pendulum,

$$H\colon T^*S^1 \to \mathbb{R},$$

is given by

$$H(q,p) = \frac{p^2}{2mr^2} - mG\cos q.$$

4. Work out Hamilton's equations for the pendulum and show that

$$\dot{q} = \frac{p}{mr^2},$$

$$\dot{p} = -mG\sin q,$$

and thus

$$\ddot{q} = -\frac{G}{r^2}\sin q.$$

Digression 1: Note that $\dot{q}$ is really the **angular velocity** *of the pendulum, while $p = mr^2\dot{q}$ is really its* **angular momentum***.*

Digression 2: If the angle q stays small, we can use the approximation $\sin q \simeq q$ to approximate the pendulum by a harmonic oscillator with

$$\ddot{q} = -\frac{G}{r^2}q.$$

However, when the angle becomes large the pendulum becomes very different from the harmonic oscillator. For example, if the pendulum starts out at $q = \pi, p = 0$ at time zero, it will stay there for all times, balanced upside down! This is an unstable equilibrium.

5. Plot the level curves of H as a function of $(q,p) \in (\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}$. Since energy is conserved, the state $(q(t), p(t))$ must stay on one of these level curves as it evolves in time. Use this to qualitatively

describe the behavior of the pendulum for various different values of the energy. In particular, find stable and unstable equilibria.

6. Supposing that the pendulum's energy equals $E \in \mathbb{R}$, show that

$$\dot{q} = \pm\sqrt{\frac{2}{mr^2}(E + mG\cos q)}. \tag{1}$$

7. To reduce the clutter and focus on essentials, switch to units where $mr^2 = mG = 1$. Using equation (1), and taking the positive square root, show that

$$t = \int \frac{dq}{\sqrt{2(E+\cos q)}}. \tag{2}$$

8. If we could do the integral in equation (2), we'd know t as a function of q. Then we could solve for q as a function of t and we'd be done! Unfortunately, this integral cannot be done using elementary functions — it's a so-called **elliptic integral**. To bring it into Jacobi's favorite form, let's work not with q but with

$$x = \sqrt{\frac{2}{E+1}}\ \sin(q/2)$$

Show that

$$\dot{x} = \sqrt{(1-x^2)(1-k^2x^2)}$$

where k, the so-called **modulus**, is given by

$$k = \sqrt{\frac{E+1}{2}}.$$

Conclude that

$$t = \int \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \tag{3}$$

This is **Jacobi's elliptic integral of the first kind**. When we solve for x as a function of t, we get an **elliptic function**.

Digression: To do integrals involving the square root of a quadratic function of x, you need inverse trig functions. However, for integrals involving the square root of a cubic or quartic function of x, you need inverse elliptic functions — or in other words, elliptic integrals. Why are they called 'elliptic'? Well, if you work out the circumference of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

you get $4a$ times this:

$$\int_0^1 \sqrt{\frac{1-k^2x^2}{1-x^2}}\, dx$$

where $k^2 = 1 - b^2/a^2$. The stuff under the square root here is not a quartic in x, but the integral is closely related to the one we've been discussing: it's called **Jacobi's elliptic integral of the second kind**.

9. I've been a bit sloppy about the limits of integration in equation (3). Show that if we start our clock so that $t = 0$ when our pendulum happens to be pointing straight down, we have

$$t = \int_0^x \frac{dy}{\sqrt{(1-y^2)(1-k^2y^2)}}.$$

If we now solve this for x as a function of t we get, by definition, the elliptic function $\mathrm{sn}(t,k)$. In other words:

$$x = \mathrm{sn}(t,k) \quad \text{means} \quad t = \int_0^x \frac{dy}{\sqrt{(1-y^2)(1-k^2y^2)}}$$

To make this more precise we'd need to worry about the branch points in the integrand at $y = \pm 1$, $y = \pm 1/k$, but let's not worry about those just yet.

Digression 3: It's easy to see that when $k = 0$, the function $\mathrm{sn}(t,k)$ reduces to the good old sine function. There is also an elliptic function $\mathrm{cn}(t,k)$ that reduces to $\cos t$ when $k = 0$, and one called $\mathrm{dn}(t,k)$ that reduces to 1 when $k = 0$. They satisfy identities like

$$\mathrm{sn}^2(t,k) + \mathrm{cn}^2(t,k) = 1 \qquad k^2\mathrm{sn}^2(t,k) + \mathrm{dn}^2(t,k) = 1$$

$$\frac{d}{dt}\mathrm{sn}(t,k) = \mathrm{cn}(t,k)\mathrm{dn}(t,k) \qquad \frac{d}{dt}\mathrm{cn}(t,k) = -\mathrm{sn}(t,k)\mathrm{dn}(t,k) \qquad \frac{d}{dt}\mathrm{dn}(t,k) = -k^2\mathrm{sn}(t,k)\mathrm{cn}(t,k),$$

so before you know it, you've got a whole world of generalized trig formulas on your hands! Back in the 1800s, any decent mathematician would know this stuff. You should too.

But now for the really cool part:

10. Using part 5, show that $q(t)$ and thus $\mathrm{sn}(t,k)$ is periodic as a function of t.

11. Show that making the replacement

$$t \mapsto it$$

in Newton's law is equivalent to reversing the sign of all forces.

In the present problem, this amounts to reversing the force of gravity, making it pull the pendulum up. But an upside-down pendulum is just another pendulum. Therefore the function $\mathrm{sn}(it,k)$ must also be periodic as a function of t. This suggests that $\mathrm{sn}(z,k)$, as a function of $z \in \mathbb{C}$, is periodic in both the real and imaginary directions. And it's true!

So, the pendulum gives a physical explanation of the fact that elliptic functions are periodic in two directions on the complex plane!

12. Prove, as rigorously as you can, that $\mathrm{sn}(z,k)$ is periodic in two directions. You can do this either by fleshing out the above argument, or by studying the integral in equation (3) and worrying about those branch points. In fact we have

$$\mathrm{sn}(z + 4K, k) = \mathrm{sn}(z,k), \qquad \mathrm{sn}(z + 2iK', k) = \mathrm{sn}(z,k)$$

where for $0 < k < 1$

$$K = \int_0^1 \frac{dy}{\sqrt{(1-y^2)(1-k^2y^2)}}$$

and

$$K' = \int_1^{1/k} \frac{dy}{\sqrt{(y^2-1)(1-k^2y^2)}}.$$