1 The position of the mass at the end of the rod is $(r \sin q, -r \cos q)$ in the x, z plane, so the speed is

$$\|(r\dot{q} \cos q, r\dot{q} \sin q)\| = r|\dot{q}|.$$

Thus, the kinetic energy is $mr^2\dot{q}^2/2$, as desired.

2 If the force is $-mG\hat{\mathbf{z}}$, then I can take the potential energy to be $mGz = -mGr \cos q$, not the claimed $-mG \cos q$ (which has the wrong units anyway).

3 The kinetic energy defines a quadratic form on $T_{q(t)}S^1 \cong \mathbb{R}$ given by the single coordinate $mr^2/2$. Thus the corresponding quadratic form on $T^*_{q(t)}S^1 \cong \mathbb{R}$ is given by the single coordinate $1/2mr^2$. Thus, the Hamiltonian is $p^2/2mr^2 - mGr \cos q$. (Note again the missing r, needed to have an energy).

4 According to Hamilton's equations, $\dot{q} = \partial H/\partial p = p/mr^2$, and $\dot{p} = -\partial H/\partial q = -mGr \sin q$. Thus, $\ddot{q} = \dot{p}/mr^2 = -G \sin q/r$. (Again, factors of r require adjustment.)

5 On a given trajectory, $H(q, p)$ takes some constant value E. Since $\cos q \leq 1$, I have

$$p^2 = 2mr^2(E + mGr \cos q) \leq 2mr^2(E + mGr),$$

or

$$-r\sqrt{2m(E + mGr)} \leq p \leq r\sqrt{2m(E + mGr)}.$$

Note that in order for this to make sense, I must have $E \geq -mGr$, so energies strtictly less than $-mGr$ are inaccessible. (This makes sense physically, since $-mGr$ is the potential energy when the mass is at its lowest position.) Similarly, since $\cos q \geq -1$, I have $p^2 \geq 2mr^2(E - mGr)$. Thus, $p = 0$ will never occur when $E > mGr$. Remembering that $E < -mGr$ is impossible, then, there are 2 broad types of motion and 2 degenerate energies. When $|E| < mGr$, the motion will pass through $p = 0$ but will never reach $\cos q = -1$. Physically, this corresponds to an oscillation about the bottom position. When $E > mGr$, in contrast, the motion will pass through $\cos q = -1$ but will never reach $p = 0$. Physically, this corresponds to traversing the entire configuration space. One degenerate energy is $E = -mGr$, where the motion remains at $\cos q = 1$ and $p = 0$, a stable equilibrium. Physically, this corresponds to remaining in the bottom position. The other degenerate energy is $E = mGr$. In this case, we could have $p = 0$ and $\cos q = -1$, another equilibrium. However, this is not required; we could have $p > 0$ and $\cos q > -1$, or alternatively $p < 0$ and $\cos q > -1$. This trajectory would approach the above equilibrium but never reach it. Therefore, this equilibrium is unstable. Physically, the unstable equilibrium corresponds to balancing at the top position, as indicated in digression 2, and the degenerate trajectory approaching this equilibrium corresponds to ever more slowly climbing to the top. A graph illustrating this is on the next page.

6 As already noted, $p^2 = 2mr^2(E + mGr \cos q)$. Since $\dot{q} = p/mr^2$, it follows that $\dot{q}^2 = 2(E + mGr \cos q)/mr^2$, so $\dot{q} = \pm\sqrt{2(E + mGr \cos q)/mr^2}$, which is equation (1) with the correct factors of r.

7 Of course, we really want $mGr = 1$, not so much $mG = 1$. (Note that the product m^2r^2G of these units gives a scale for p^2, as can be seen on the graph for problem 5, while the quotient G/r gives a scale for t^2.) Then the previous problem gives $\dot{q}^2 = 2(E + \cos q)$. Since $dt = dq/\dot{q}$, equation (2) follows if I use the principal square root. Note that this will break down when p changes sign, which can happen when $|E| < 1$ in the new units (the oscillating case).

8 The minimum value for the energy is -1 in these new units, so the formula offered for the modulus k must make sense. Since $x = \sin(q/2)/k$, I have $\dot{x} = \dot{q}\cos(q/2)/2k$, so

$$\dot{x}^2 = \dot{q}^2\cos^2(q/2)/4k^2 = (E + \cos q)(1 + \cos q)/4k^2 = (2k^2 - 1 + \cos q)(1 + \cos q)/4k^2.$$

Meanwhile, $x^2 = (1 - \cos q)/2k^2$, so $2k^2 - 1 + \cos q = 2k^2(1 - x^2)$, and $1 + \cos q = 2(1 - k^2x^2)$. Thus, $\dot{x}^2 = (1 - x^2)(1 - k^2x^2)$. Now, in the oscillating case, our analysis will already break down at $\cos q = -E$, or $1 - x^2 = 0$; in the case of perpetual rotation, on the other hand, $1 - x^2$ is always strictly positive. In that case, furthermore, $\dot{x}$ will remain nonnegative so long as $\cos(q/2)$ does, a simplification that breaks down at $\cos q = -1$, or $1 - k^2x^2 = 0$; but in the oscillating case, that is never reached. Thus in either case, the formula $\dot{x} = \sqrt{(1 - x^2)(1 - k^2x^2)}$ will be valid on any interval containing $x = 0$ where it makes sense. (The degenerate nonequilibrium will find this formula valid and true always, while the equilibria will find the formula never defined.) Formula (3) then follows, so long as it is applied on an interval containing 0.

9 Here I indeed integrate on an interval containing 0. The correct form may vary from that given by a constant, but since it gives the correct value $t = 0$ for $x = 0$, it must be exactly right. We can now define the elliptic function sn for all values of x by making recourse to the physical problem of the pendulum, since that definition agrees with the integral whenever the latter makes sense.