The Kepler Problem Revisited:
The Laplace–Runge–Lenz Vector
Classical Mechanics Homework
March 17, 2∞8
John Baez homework by C.Pro

Whenever we have two particles interacting by a central force in 3d Euclidean space, we have conservation of energy, momentum, and angular momentum. However, when the force is gravity — or more precisely, whenever the force goes like $1/r^2$ — there is an extra conserved quantity. This is often called the **Runge–Lenz vector**, but it was originally discovered by Laplace. Its existence can be seeen in the fact that in the gravitational 2-body problem, each particle orbits the center of mass in an ellipse (or parabola, or hyperbola) whose *perihelion does not change with time*. The Runge–Lenz vector points in the direction of the perihelion! If the force went like $1/r^{2.1}$, or something like that, the orbit could still be roughly elliptical, but the perihelion would 'precess' — that is, move around in circles.

Indeed, the first piece of experimental evidence that Newtonian gravity was not quite correct was the precession of the perihelion of Mercury. Most of this precession is due to the pull of other planets and other effects, but about 43 arcseconds per century remained unexplained until Einstein invented general relativity.

In fact, we can use the Runge–Lenz vector to simplify the proof that gravitational 2-body problem gives motion in ellipses, hyperbolas or parabolas. Here's how it goes. As before, let's work with the relative position vector

$$\mathbf{q}(t) = \mathbf{q}_1(t) - \mathbf{q}_2(t)$$

where $\mathbf{q}_1, \mathbf{q}_2 \colon \mathbb{R} \to \mathbb{R}^3$ are the positions of the two bodies as a function of time. In what follows we will use q to stand for the magnitude of the vector $\mathbf{q}$, and $\hat{\mathbf{q}}$ to stand for a normalized vector that points in the direction of $\mathbf{q}$:

$$q = |\mathbf{q}|, \qquad \hat{\mathbf{q}} = \frac{\mathbf{q}}{q}.$$

In a previous homework we saw that $\mathbf{q}(t)$ satisfies

$$m\ddot{\mathbf{q}} = f(q)\hat{\mathbf{q}}$$

where f is the force as a function of distance, and m is the so-called 'reduced mass'. Since the force of gravity goes like $1/r^2$, we have

$$f(q) = -k/q^2$$

where k is the same constant as in the previous homework. We thus have

$$m\ddot{\mathbf{q}} = -k\hat{\mathbf{q}}/q^2 \tag{1}$$

and this will be our starting-point for all that follows.

1. Define the angular momentum vector $\mathbf{J}$ in the usual way:

$$\mathbf{J} = m\mathbf{q} \times \dot{\mathbf{q}}$$

We know from our previous work that angular momentum is conserved:

$$\dot{\mathbf{J}} = 0.$$

Take the cross product of both sides of equation (1) with the vector $\mathbf{J}$. Simplify the right-hand side and show that

$$\ddot{\mathbf{q}} \times \mathbf{J} = k\dot{\hat{\mathbf{q}}} \tag{2}$$

Solution: First we compute the time derivative of $\hat{\mathbf{q}}$. Let $\mathbf{q}_i/q$ be the i^{th} component of $\hat{\mathbf{q}}$ and write $q = \sqrt{\mathbf{q}\cdot\mathbf{q}}$. Then use the quotient rule to obtain

$$\begin{aligned} \frac{d}{dt}\frac{\mathbf{q}_i}{q} &= \frac{\dot{\mathbf{q}}_i q - \mathbf{q}_i\left(\frac{\mathbf{q}\cdot\dot{\mathbf{q}}}{q}\right)}{q^2} \\ &= \frac{\dot{\mathbf{q}}_i(\mathbf{q}\cdot\mathbf{q}) - (\mathbf{q}\cdot\dot{\mathbf{q}})\cdot\mathbf{q}_i}{q^3} \end{aligned}$$

and so,

$$\dot{\hat{\mathbf{q}}} = \frac{\dot{\mathbf{q}}(\mathbf{q}\cdot\mathbf{q}) - (\mathbf{q}\cdot\dot{\mathbf{q}})\cdot\mathbf{q}}{q^3}$$

Now, using the identity

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c},$$

we have

$$\begin{aligned} m\ddot{\mathbf{q}}\times J &= -mk\hat{\mathbf{q}}/q^2 \times (m\mathbf{q}\times\dot{\mathbf{q}}) \\ &= \frac{mk}{q^3}\left((\mathbf{q}\cdot\mathbf{q})\dot{\mathbf{q}} - (\mathbf{q}\cdot\dot{\mathbf{q}})\mathbf{q}\right) \\ &= mk\dot{\hat{\mathbf{q}}}, \end{aligned}$$

and dividing both sides by m we obtain the result.

2. Use parts 1 to show that

$$\frac{d}{dt}(\dot{\mathbf{q}}\times\mathbf{J}) = k\dot{\hat{\mathbf{q}}}.$$

Solution :

$$\begin{aligned} \frac{d}{dt}(\dot{\mathbf{q}}\times\mathbf{J}) &= \ddot{\mathbf{q}}\times\mathbf{J} + \mathbf{q}\times\dot{\mathbf{J}} \\ &= k\dot{\hat{\mathbf{q}}} + \mathbf{q}\times 0 \\ &= k\dot{\hat{\mathbf{q}}}. \end{aligned}$$

3. Use part 2 to show that

$$\dot{\mathbf{q}}\times\mathbf{J} = k\hat{\mathbf{q}} + \mathbf{x}$$

for some vector $\mathbf{x}\in\mathbb{R}^3$ that is *independent of time.*

Solution: Note 2 says

$$\frac{d}{dt}((\dot{\mathbf{q}}\times\mathbf{J}) - k\hat{\mathbf{q}}) = 0$$

and so the two functions must only differ by some constant vector, say x.

It will be handy to divide this vector by k, obtaining the **Runge–Lenz vector**:

$$\mathbf{A} = \frac{\mathbf{x}}{k}$$

which clearly is also independent of time. In other words, the Runge–Lenz vector

$$\mathbf{A} = \frac{\dot{\mathbf{q}}\times\mathbf{J}}{k} - \hat{\mathbf{q}} \tag{3}$$

is a conserved quantity for the Kepler problem:

$$\dot{\mathbf{A}} = 0.$$

4. Use equation (3) to show that

$$\mathbf{A} \cdot \mathbf{q} = \frac{\mathbf{J} \cdot \mathbf{J}}{km} - q. \tag{4}$$

Solution: Using the identity $(a \times b) \cdot c = (c \times a) \cdot b$ we have

$$\begin{aligned} \mathbf{A} \cdot \mathbf{q} &= \frac{\dot{\mathbf{q}} \times \mathbf{J}}{k} \cdot \left(\frac{m\mathbf{q}}{m}\right) - \hat{\mathbf{q}} \cdot \mathbf{q} \\ &= \frac{(m\mathbf{q} \times \dot{\mathbf{q}}) \cdot J}{mk} - \frac{q^2}{q} \\ &= \frac{J \cdot J}{mk} - q \end{aligned}$$

as desired.

5. Now write

$$\mathbf{A} \cdot \mathbf{q} = Aq \cos\theta$$

where $A = |\mathbf{A}|$ is the magnitude of the Runge–Lenz vector and θ is the angle between $\mathbf{q}$ and $\mathbf{A}$. Using equation (4), show that

$$q = \frac{\mathbf{J} \cdot \mathbf{J}}{km} \frac{1}{1 + A\cos\theta}.$$

Solution: From equation (4)

$$\mathbf{A} \cdot \mathbf{q} + q = \frac{\mathbf{J} \cdot \mathbf{J}}{km}$$

so

$$q(A\cos\theta + 1) = \frac{\mathbf{J} \cdot \mathbf{J}}{km}$$

and therefore

$$q = \frac{\mathbf{J} \cdot \mathbf{J}}{km} \frac{1}{1 + A\cos\theta}.$$

Now, let me explain what you have achieved!

The above equation looks almost like this equation in our previous homework about the Kepler problem:

$$r = \frac{p}{1 + e\cos(\theta - \theta_0)}.$$

And indeed, they are really just different ways of writing the same equation, except that now we have rotated our polar coordinate system so that $\theta_0 = 0$. Another way of saying this is that in these coordinates, θ is zero at the perihelion of the orbit.

So, you've just given a new proof that the orbit in the Kepler problem must be an ellipse, parabola or hyperbola! Moreover, comparing the two equations above we see that

$$e = A,$$

so *the magnitude of the Runge–Lenz vector is the eccentricity of the orbit.* We also see that *the Runge–Lenz vector points in the direction of the orbit's perihelion.* Thus the conservation of the

Runge–Lenz vector is just a way of saying the eccentricity and perihelion don't change with time! Finally, we see that

$$p = \frac{\mathbf{J} \cdot \mathbf{J}}{km}$$

which we already saw last time.

The story of the Runge–Lenz vector goes much deeper than this, and I encourage you to look at my webpage about it, and also the excellent Wikipedia article:

- John Baez, The mysteries of the gravitational 2-body problem, http://math.ucr.edu/home/baez/gravitational.html
- Wikipedia, Laplace–Runge–Lenz vector, http://en.wikipedia.org/wiki/Laplace-Runge-Lenz_vector

Here's the story in a nutshell. First, the Runge–Lenz vector is conserved not only in the classical, but also in the *quantum-mechanical* theory of two particles attracted by an inverse square force law. The main example is the *hydrogen atom*. The surprising degeneracy of energy levels for the hydrogen atom, also important for the periodic table of elements, is special to an inverse square force law and due to the conservation of the Runge–Lenz vector.

Mathematically, conservation of angular momentum is due to rotation symmetry: symmetry under the group SO(3). Any central force has this symmetry. But the inverse square force law actually has symmetry under a bigger group, SO(4) or SO(3, 1). SO(4) is the rotational group in 4 dimensions, while SO(3, 1) is the Lorentz group. These are both 6-dimensional groups, so they give 6 conserved quantities: the 3 components of $\mathbf{J}$ and the three components of $\mathbf{K}$, which is a rescaled version of the Runge–Lenz vector:

$$\mathbf{K} = k\sqrt{\frac{m}{2|H|}}\mathbf{A}$$

where H is the Hamiltonian. This new vector $\mathbf{K}$ is conserved because $\mathbf{A}$ and H are.

To see this in detail, one can calculate that

$$\{J_1, J_2\} = J_3 \qquad \text{and cyclic permutations}$$

$$\{J_1, K_2\} = K_3 \qquad \text{and cyclic permutations}$$

and, oddly,

$$\{K_1, K_2\} = \pm K_3 \qquad \text{and cyclic permutations}$$

where the sign is always $+$ if the energy is negative, and $-$ if the energy is positive. (Here by 'and cyclic permutations', I mean we can cyclically permute the indices $1, 2, 3$ and get other true equations.)

This means that if we restrict to

$$X_- = \{(q,p) \in \mathbb{R}^3 \times \mathbb{R}^3 \colon H(q,p) < 0\}$$

we get the above equations with the plus sign, and these are precisely the formulas for the Lie bracket in $\mathfrak{so}(4)$. If we restrict to

$$X_+ = \{(q,p) \in \mathbb{R}^3 \times \mathbb{R}^3 \colon H(q,p) < 0\}$$

we get the above equations with the minus sign, which are the formulas for the Lie bracket in $\mathfrak{so}(3,1)$.

The spaces X_- and X_+ are Poisson manifolds in their own right. X_- is the phase space for particles in elliptical orbits, while X_+ is the phase space for particles in hyperbolic orbits. With some work, one can show that $\mathrm{SO}(4)$ acts on X_- and $\mathrm{SO}(3,1)$ acts on X_+. Both actions are Hamiltonian, and the corresponding observables are conserved quantities: just $\mathbf{J}$ and $\mathbf{K}$.

In short: *the problem of a particle in an elliptical orbit in gravity has* $\mathrm{SO}(4)$ *symmetry!* But *why?* What does gravity have to do with 4d rotations? An explanation was given by the Russian physicist Fock in 1935.

Fock showed that X_- is isomorphic, as a Poisson manifold, to T^*S^3 — the phase space for a particle on the sphere $S^3 \subseteq \mathbb{R}^4$. Even better, the whole problem of a particle in an elliptical orbit in an inverse-square force law is *isomorphic to the problem of a free particle on* S^3*!* The latter problem has an *obvious* symmetry under $\mathrm{SO}(4)$.

Of course, all this is puzzling in its own right. It goes to show that even the most classic of classical mechanics problems still holds mysteries! Here is a good book on the subject:

- Victor Guillemin and Shlomo Sternberg, *Variations on a Theme by Kepler*, American Mathematical Society, Providence, Rhode Island, 1990.