

Classical Mechanics, Lecture 13

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1 Lie Algebras

Lie groups are groups that are also manifolds. These let us describe continuous symmetries in physics. A classic example is $SO(n)$ - the group of rotations of \mathbb{R}^n . Every Lie group has a “Lie algebra”, which lets us do calculations more easily. The Lie algebra is a vector space, so it lets us use linear algebra to study Lie groups.

For example: $SO(n)$ has a Lie algebra

$$\mathfrak{so}(n) = \{A : A \text{ is an } n \times n \text{ real matrix with } A^* = -A\}$$

The point is that given $A \in \mathfrak{so}(n)$,

$$\exp(tA) = \sum_{k=0}^{\infty} \frac{(tA)^k}{k!} \in SO(n), \quad (t \in \mathbb{R})$$

In fact, any Lie group G has a Lie algebra \mathfrak{g} and there is a map called exponentiation

$$\exp: \mathfrak{g} \rightarrow G$$

such that

$$\exp((s+t)A) = \exp(sA) \exp(tA).$$

For example, if $\mathfrak{g} = \mathbb{R}$ (a vector space) and $G = \mathbb{R}^+$ (the Lie group of positive real numbers, with multiplication as the group operation), then

$$\exp: \mathbb{R} \rightarrow \mathbb{R}^+$$

is the usual exponential function and I am just saying $\exp(s+t) = \exp(s)\exp(t)$. Here a slide rule (or table of logarithms) converts problems in the Lie group into problems in the Lie algebra.

Definition 1 A Lie algebra is a (real) vector space \mathfrak{g} equipped with an operation

$$[\cdot, \cdot]: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$$

called the **Lie bracket** which has the following properties:

1. bilinearity
2. antisymmetry
3. the Jacobi identity

We’ve seen one type of Lie algebra so far: if X is a Poisson manifold, $C^\infty(X)$ is a Lie algebra with the Poisson bracket as its bracket. But in fact, every manifold gives a Lie algebra, in a different way:

Theorem 2 Given any manifold X , let $\text{Vect}(X)$ be the set of vector fields on X . This becomes a Lie algebra by:

$$\begin{aligned}(\alpha v)f &= \alpha(v(f)) \\ (v+w)f &= vf + wf \\ [v,w]f &= v(wf) - w(vf)\end{aligned}$$

where $\alpha \in \mathbb{R}$ and $f \in C^\infty(X)$.

Sketch of proof:

Just calculate to check the Lie algebra axioms. For example: a vector field $v \in \text{Vect}(X)$ is a linear map

$$v: C^\infty(X) \rightarrow C^\infty(X)$$

satisfying $v(fg) = v(f)g + fv(g)$.

Why does $v, w \in \text{Vect}(X) \Rightarrow [v, w] \in \text{Vect}(X)$?

$$\begin{aligned}[v, w](fg) &= vw(fg) - wv(fg) \\ &= v(w(f)g + fw(g)) - w(v(f)g + fv(g)) \\ &= (vw(f))g + w(f)v(g) + v(f)w(g) + f(vw(g)) - wv(f)g - v(f)w(g) - w(f)v(g) - f(wv(g)) \\ &= ([v, w]f)g + f([v, w]g).\end{aligned}$$

Why does $[\cdot, \cdot]$ satisfy the Jacobi identity?

$$\begin{aligned}[u, [v, w]] &= [[u, v], w] + [v, [u, w]] \\ ([[u, v], w] + [v, [u, w]])f &= (uvw - vuw - wuv + wvu + vuw - vwu - uwv + uwv)f \\ &= [u, [v, w]]f\end{aligned}$$

Etc...

This Lie algebra $\text{Vect}(X)$ is very related to the Poisson algebra $C^\infty(X)$ when X is a Poisson manifold:

Theorem 3 If X is a Poisson manifold and $f, g \in C^\infty(X)$ then

$$[v_f, v_g] = v_{\{f, g\}}.$$

Proof - For any $h \in C^\infty(X)$,

$$\begin{aligned}v_{\{f, g\}}h &= \{\{f, g\}, h\} \\ &= \{f\{g, h\}\} + \{\{f, h\}, g\} \\ &= \{f, \{g, h\}\} - \{g, \{f, h\}\} \\ &= v_f v_g h - v_g v_f h \\ &= [v_f, v_g]h\end{aligned}$$

Later we'll define a 'homomorphism' between Lie algebras, and the theorem we just proved will say that

$$v \mapsto v_f$$

is a homomorphism from the Lie algebra $C^\infty(X)$ to the Lie algebra $\text{Vect}(X)$.