Connections as Functors

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In this homework we'll see how a vector bundle E equipped with a connection over a manifold X gives a functor

$$F\colon \mathcal{P}(X) \to \mathrm{Vect}$$

where $\mathcal{P}(X)$ is the category of paths in X, defined in the last homework. Recall that objects of $\mathcal{P}(X)$ are points of X, while morphisms are piecewise-smooth paths in X. The functor F maps each point $x \in X$ to the the fiber of E over the point x. Similarly, F maps each path $\gamma\colon x \to y$ in X to a linear operator

$$F(\gamma)\colon F(x) \to F(y)$$

defined using parallel transport along the path γ.

To warm up, let's see how any linear ordinary differential equation gives a functor. I'll let you use this fact:

Theorem 1. *Let $\mathrm{End}(\mathbb{R}^n)$ be the algebra of linear operators from $\mathbb{R}^n$ to itself. Suppose $A\colon [a,b] \to \mathrm{End}(\mathbb{R}^n)$ is any smooth function and $t_0 \in [a,b]$. Given any vector $\psi_0 \in V$, the differential equation*

$$\frac{d\psi(t)}{dt} = A(t)\psi(t) \tag{1}$$

has a unique smooth solution $\psi\colon [a,b] \to \mathbb{R}^n$ with $\psi(t_0) = \psi_0$.

Sketch of Proof. *With the given initial conditions, Equation (1) is equivalent to the integral equation*

$$\psi(t) = \psi_0 + \int_{t_0}^{t} A(s)\psi(s)ds.$$

A solution of this equation is none other than a fixed point of the map T sending the function ψ to the function $T\psi$ given by

$$(T\psi)(t) = \psi_0 + \int_{t_0}^{t} A(s)\psi(s)ds.$$

T maps the Banach space of continuous $\mathbb{R}^n$-valued functions on $[a,b]$ to itself. If $\int_a^b \|A(s)\|ds = M$ then

$$\|T(\psi_1) - T(\psi_2)\| \le M\|\psi_1 - \psi_2\|.$$

We call a map with this property a **contraction** *if $M < 1$. An easy argument shows that any contraction on a Banach space has a unique fixed point, so our equation has a unique solution. If $c \not< 1$, we can chop the interval $[a,b]$ into smaller intervals for which this bound does hold, and prove the theorem one piece at a time.* $\square$

1. Let $K[a,b]$ be the category whose objects are points of the interval $[a,b]$, with exactly one morphism from any object to any other. Given a function $A\colon [a,b] \to \mathrm{End}(\mathbb{R}^n)$ satisfying the conditions of Theorem 1, use the theorem to prove there is a unique functor

$$F\colon K[a,b] \to \mathrm{Vect}$$

such that:

- F sends any object to $\mathbb{R}^n$.

- F sends any morphism $f\colon t_0 \to t_1$ to the linear operator

$$\psi_0 \mapsto \psi(t_1)$$

  where $\psi\colon [a,b] \to \mathbb{R}^n$ is the unique solution of Equation (1) with $\psi(t_0) = \psi_0$.

More poetically, $K[a,b]$ is the category whose objects are **moments of time** *between time a and time b. The morphisms in this category are* **passages of time**. *Applied to the passage of time from t_0 to t_1, the functor F gives the* **time evolution operator** *mapping $\psi(t_0)$ to $\psi(t_1)$, where ψ is any solution of*

$$\frac{d\psi(t)}{dt} = A(t)\psi(t).$$

Next, suppose X is a smooth manifold and A is a smooth $\mathrm{End}(\mathbb{R}^n)$*-valued 1-form on X. For each point $x \in X$, such a thing gives a linear map*

$$A_x\colon T_xX \to \mathrm{End}(\mathbb{R}^n),$$

and A_x varies smoothly as a function of x. If we take $n = 1$, A becomes an ordinary 1-form and the following result reduces to a problem in the last homework assignment:

2. Suppose A is a smooth $\mathrm{End}(\mathbb{R}^n)$-valued 1-form on the manifold X. Show that there is a unique functor

$$F\colon \mathcal{P}(X) \to \mathrm{Vect}$$

such that

- F sends any point of X to $\mathbb{R}^n$.
- F sends any piecewise-smooth path $\gamma\colon [0,T] \to X$ to the linear operator

$$\psi_0 \mapsto \psi(T)$$

  where $\psi\colon [0,T] \to \mathbb{R}^n$ is the unique solution of the equation

$$\frac{d\psi(t)}{dt} = A_{\gamma(t)}(\gamma'(t))\, \psi(t) \tag{2}$$

  with $\psi(0) = \psi_0$.

An $\mathrm{End}(\mathbb{R}^n)$*-valued 1-form A is called a* **connection** *on the trivial vector bundle*

$$\pi\colon X \times \mathbb{R}^n \to X.$$

If $\psi(t)$ satisfies Equation (2), we say the vector $\psi(t)$ is **parallel transported** *along the path γ using the connection A. The linear operator $F(\gamma)$ is called the* **holonomy** *of the connection A along the path γ.*

All this stuff generalizes to the case of a connection on a nontrivial vector bundle

$$\pi\colon E \to X$$

except that now the functor F maps each point $x \in X$ to the **fiber of E over** *x, namely $E_x = \pi^{-1}(x)$. To handle this case, we choose an open cover of X such that E restricted to each open set is trivializable, and reduce the problem to the case treated above. After huffing and puffing, we get:*

Theorem 2. *Suppose A is a smooth connection on a smooth vector bundle $\pi: E \to X$ over a smooth manifold X. Then there is a unique functor*

$$F: \mathcal{P}(X) \to \mathrm{Vect}$$

such that:

- *For any object x of $\mathcal{P}(X)$, $F(x)$ is the fiber of E over x.*
- *For any morphism $\gamma: x \to y$ of $\mathcal{P}(X)$, $F(\gamma)$ is the holonomy of A along γ.*

The converse is not true: there are functors $F: \mathcal{P}(X) \to \mathrm{Vect}$ that don't come from connections on vector bundles! However, we can characterize the functors that do by means of three conditions:

- $F(\gamma_1) = F(\gamma_2)$ *when γ_2 is obtained by reparametrizing γ_1:*

  $$\gamma_2(t) = \gamma_1(f(t))$$

  for any monotone increasing function f.
- $F(\gamma_2) = F(\gamma_1)^{-1}$ *when γ_2 is a reversed version of γ_1:*

  $$\gamma_2(t) = \gamma_1(f(t))$$

  for any monotone decreasing function f.
- $F(\gamma)$ *depends smoothly on γ (in a certain precise sense).*

For some hints on how to prove this, try:

> J. Barrett, Holonomy and path structures in general relativity and Yang–Mills theory, *Int. J. Theor. Phys.*, **30** (1991), 1171–1215.

If we drop the smoothness condition, we call F a **generalized connection**. *These play an important role in loop quantum gravity.*