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John Baez <johnb@ucr.edu>

endomorphism rings of abelian varieties

13 messages

John Baez <john.baez@ucr.edu> Thu, Feb 24, 2022 at 8:11 PM
Reply-To: baez@math.ucr.edu
To: JAMES DOLAN <james.dolan1@students.mq.edu.au>

Hi -

I don't know how much you want to keep digging into abelian varieties versus expanding out to bigger pictures, but I've been reading some neat stuff about abelian varieties.

First, if X is a complex torus we can form End(X), the set of holomorphic endo-maps that are also group homomorphisms. And this set is a ring for general abstract nonsense reasons, with composition as the multiplication.

People study these rings a lot. They often tensor them with the rationals to make them more tractable. Then you can show this rationalized End(X) is a semisimple algebra.

There's nice stuff to say about the general case where End(X) is a direct product of simple factors: it gives you a way to write X as a product of so-called "simple" abelian varieties, up to isogeny. A "simple" abelian variety is one that doesn't have any abelian subvarieties except for {0} and X.

But apparently where this stuff really shines is in describing and computing the Neron-Severi group. I won't describe that now because it takes a while and now it's dinner time! But it seems pretty cool.

(One "downside" is that to get really deep you need to understand division algebras over the rationals, since simple algebras over some field are matrix algebras with entries in some division algebra over Q. But this must have its own charm.)

Best,
jb

JAMES DOLAN <james.dolan1@students.mq.edu.au> Fri, Feb 25, 2022 at 11:46 AM
To: John Baez <baez@math.ucr.edu>

"I don't know how much you want to keep digging into abelian varieties versus expanding out to bigger pictures, but"

it's actually easy to come up with lots of good excuses for examining abelian varieties in lots of detail, for example along the lines of that wikipedia bit about:

"The study of differential forms on C, which give rise to the abelian integrals with which the theory started, can be derived from the simpler, translation-invariant theory of differentials on J. The abelian variety J is called the Jacobian variety of C, for any non-singular curve C over the complex numbers."

or perhaps trying to re-express approximately the same idea in more "modern" language, that by studying abelian varieties we're studying some of the "motivic content" of more general varieties, and preparing to understand more general kinds of such "motivic content" .... to the extent that that makes any sense ....

"I've been reading some neat stuff about abelian varieties."

this all sounds very interesting, and a lot of it sounds like it's dealing with what i was calling "complex multiplication (on abelian varieties)" .... i have to think more about to what extent it's going beyond that .... also about to what extent the alleged interaction with neron-severi lattices connects with stuff i think i understand (on a somewhat mystical intuitive level ....) about (generalized ....) "jugendtraum" ideas ....
[Quoted text hidden]

John Baez <john.baez@ucr.edu> Fri, Feb 25, 2022 at 5:44 PM
Reply-To: baez@math.ucr.edu
To: JAMES DOLAN <james.dolan1@students.mq.edu.au>

Hi -

I wrote:

> First, if X is a complex torus we can form End(X), the set of holomorphic endo-maps that are also group homomorphisms. And this set is a ring for general abstract nonsense reasons, with composition as the multiplication.
>
> People study these rings a lot. They often tensor them with the rationals to make them more tractable. Then you can show this rationalized End(X) is a semisimple algebra.

I'm pretty sure that for the cases we've been talking about, where X is a product of two copies of an imaginary quadratic number ring, the rationalized End(X) is a dense subring of the 2x2 complex matrices.

Let me say a bit more about stuff I'm reading:

Any polarization on X gives an algebra anti-involution called * on the rationalized End(X), called the **Rosati involution.** (This should have been the title of a Robert Ludlum book.)

I won't explain how we get this; I'll just say some good properties of it.

As usual we say an element f of the rationalized End(X) is **self-adjoint** if f = f*, and **positive** if it's of the form f = gg*.

I think End(X) is a subring of the rationalized End(X), so we can talk about self-adjoint elements of End(X). These form a group under addition.

And here's a cool fact: this group is the Neron-Severi group of X! Even better, the positive elements of End(X) are the polarizations.

Now I'm gonna make a guess:

- I bet that in our favorite cases, where X is the square of an elliptic curve with complex multiplication, End(X) is 2x2 matrices with entries in the corresponding algebraic number ring Z[sqrt(-d)].

This implies the rationalized End(X) is 2x2 matrices with entries in the number field Q[sqrt(-d)].

- I bet that self-adjoint elements of the rationalized X are just 2x2 matrices with entries in Q[sqrt(-d)] that are self-adjoint in the usual sense for complex matrices.

This implies that the Neron-Severi group of X consists of 2x2 self-adjoint matrices with entries in Z[sqrt(-d)].

So, it's a lattice in the vector space of self-adjoint 2x2 complex matrices! And this is Minkowski spacetime.

The positive elements in this lattice are the polarizations.

It's sort of funny how the Rosati involution depends on a choice of polarization. But I think this fact is actually familiar in our favorite cases! A choice of polarization gives a future-pointing time-like direction in Minkowski spacetime. Only by choosing such a direction can we make Minkowski spacetime into a Jordan algebra. This is the Jordan algebra of self-adjoint elements in a star-algebra. And that star-algebra is isomorphic to the algebra of 2x2 complex matrices.

But I guess what I'm saying is that there's no way to embed Minkowski spacetime into a star-algebra like this in a Lorentz-invariant way: doing so *requires* a choice of future-pointing direction. So that's why the Rosati involution depends on a choice of polarization... in this case!

Jordan algebras are showing up here. That's fun to see.

Best,
jb

JAMES DOLAN <james.dolan1@students.mq.edu.au> Mon, Feb 28, 2022 at 12:46 PM
To: John Baez <baez@math.ucr.edu>

"There's nice stuff to say about the general case where End(X) is a direct product of simple factors: it gives you a way to write X as a product of so-called "simple" abelian varieties, up to isogeny. A "simple" abelian variety is one that doesn't have any abelian subvarieties except for {0} and X."

ok that's very interesting .... in retrospect maybe it should have been obvious .... might be interesting in connection with trying to understand "abstract hyperellipticness" of curves ....

....
On Thu, Feb 24, 2022 at 11:11 PM John Baez <john.baez@ucr.edu> wrote:
[Quoted text hidden]

JAMES DOLAN <james.dolan1@students.mq.edu.au> Mon, Feb 28, 2022 at 12:58 PM
To: John Baez <baez@math.ucr.edu>

"I'm pretty sure that for the cases we've been talking about, where X is a product of two copies of an imaginary quadratic number ring, the rationalized End(X) is a dense subring of the 2x2 complex matrices."

i'm not sure yet what to make of the interplay between commutative and noncommutative rings of endomorphisms here .... "complex multiplication on abelian varieties" seems to be all about the commutative ones .... so far i have only vague ideas of how far you might be able to get in exploiting the noncommutative ones ....

"Any polarization on X gives an algebra anti-involution called * on the rationalized End(X), called the Rosati involution. (This should have been the title of a Robert Ludlum book.)

I won't explain how we get this; I'll just say some good properties of it."

this seems fairly straightforward in terms of the picture of "star-modules of a star-algebra" as something like inner-product spaces in which the abstract star-operator becomes concrete in terms of linear duality ....

"As usual we say an element f of the rationalized End(X) is self-adjoint if f = f*, and positive if it's of the form f = gg*.

I think End(X) is a subring of the rationalized End(X), so we can talk about self-adjoint elements of End(X). These form a group under addition.

And here's a cool fact: this group is the Neron-Severi group of X! Even better, the positive elements of End(X) are the polarizations."

i didn't get enough of a chance to think about this yet, but again it looks like it should be pretty straightforward and nice (as a tool for understanding neron-severi, in particular) ....

"Now I'm gonna make a guess:
I bet that in our favorite cases, where X is the square of an elliptic curve with complex multiplication, End(X) is 2x2 matrices with entries in the corresponding algebraic number ring Z[sqrt(-d)].
This implies the rationalized End(X) is 2x2 matrices with entries in the number field Q[sqrt(-d)].
I bet that self-adjoint elements of the rationalized X are just 2x2 matrices with entries in Q[sqrt(-d)] that are self-adjoint in the usual sense for complex matrices.
This implies that the Neron-Severi group of X consists of 2x2 self-adjoint matrices with entries in Z[sqrt(-d)].

So, it's a lattice in the vector space of self-adjoint 2x2 complex matrices! And this is Minkowski spacetime.

The positive elements in this lattice are the polarizations.

It's sort of funny how the Rosati involution depends on a choice of polarization. But I think this fact is actually familiar in our favorite cases! A choice of polarization gives a future-pointing time-like direction in Minkowski spacetime. Only by choosing such a direction can we make Minkowski spacetime into a Jordan algebra. This is the Jordan algebra of self-adjoint elements in a star-algebra. And that star-algebra is isomorphic to the algebra of 2x2 complex matrices.

But I guess what I'm saying is that there's no way to embed Minkowski spacetime into a star-algebra like this in a Lorentz-invariant way: doing so requires a choice of future-pointing direction. So that's why the Rosati involution depends on a choice of polarization... in this case!

Jordan algebras are showing up here. That's fun to see."

i still need to read (and/or talk about; perhaps this afternoon) some more!

....
[Quoted text hidden]

John Baez <john.baez@ucr.edu> Mon, Feb 28, 2022 at 11:57 PM
Reply-To: baez@math.ucr.edu
To: JAMES DOLAN <james.dolan1@students.mq.edu.au>

Hi -

It turns out my guess about Jordan algebras is not only right but known. The Wikipedia article "Rosati involution" explains how any polarization makes the rationalized Neron-Severi group into a Jordan algebra! Moreover they say it's a formally real Jordan algebra - the kind where a sum of squares only be zero if each one is zero. This is the kind of Jordan algebra that shows up in the Jordan-Wigner-von Neumann paper on the foundations of quantum mechanics: the kind that has a well-behaved concept of positive element, and gives rise to an axiomatic projective space.

So you were right when you said that when we go beyond abelian surfaces to higher-dimensional abelian varieties, the math would be more like QM than SR.

I would like to read more about this. I want to know what the Jordan product *means*. I also know tons less about formally real Jordan algebras over the rationals than over the reals, where there's a famous classification. (In the present situation I don't think we get the "weird" formally real Jordan algebras, namely the Clifford-Jordan algebras other than the 2x2 self-adjoint matrix ones, or the exceptional Jordan algebra.) I'd also like to know how far we can get without rationalizing the Neron-Severi group. The Neron-Severi group might itself become a Jordan *ring*, at least if we use a principal polarization.

Best,
jb

[Quoted text hidden]

JAMES DOLAN <james.dolan1@students.mq.edu.au> Tue, Mar 1, 2022 at 4:18 AM
To: John Baez <baez@math.ucr.edu>

"Moreover they say it's a formally real Jordan algebra - the kind where a sum of squares only be zero if each one is zero."

yes, it's clear we "need" this to get that nice cone of ample line bundles, i think ....

(hmm, now i want to dredge up that decades-(or-so)-old email conversation with that person who worked on "jordan cones" (among a scattered collection of other things that they worked on as well). (googling "jordan cone" just now did suggest various ideas ....) maybe it was michael kinyon?? not sure. there's a recurring theme about defining a cone of "positives" in terms of squaring (and/or some mutant form of "squaring" such as multiplying an element by its adjoint, which is hard to tell from squaring if everything's self-adjoint) wrt some formally real commutative (though not necessarily associative) multiplication. for example if you do this to the commutative multiplication in a formally real poisson algebra then you recover the cone of positive legendrian submanifolds in some sense (i forget the details, but it works! it's another one of those "post-graduate re-interpretation of freshman-calculus minimization-of-quadratic-polynomials techniques" ideas); furthermore you can almost recover the commutative multiplication just from knowledge of that cone, except you sort-of "lose track of the multiplicative identity-element"; in the poisson case this corresponds to the fact that you can recover a contact manifold though not a symplectic manifold just from knowing the cone of positives inside the "infinitesimal (but infinite-dimensional!) homogeneous space of legendrian submanifolds", or something like that.

anyway, if you're looking for a conceptual interpretation of the jordan product of neron-severi elements, the above means that you can try to weasel out of it by saying that the jordan product is mostly just encoding the cone of ample elements together with a choice of favorite principal polarization to serve as the jordan identity-element (i think). however you could also try to _not_ weasel out of it, in which case there are some pretty speculative possibilities to explore! like, i think i see some vague possibility here of groupoidifying the jordan product into some sort of groupoid-span involving "hecke isogenies of abelian varieties" or something like that, vaguely analogous to how the lie bracket in a "hall lie-algebra" gets groupoidified into a groupoid-span involving short exact sequences of indecomposable quiver representations .... i did say that it's speculative!)

"This is the kind of Jordan algebra that shows up in the Jordan-Wigner-von Neumann paper on the foundations of quantum mechanics: the kind that has a well-behaved concept of positive element, and gives rise to an axiomatic projective space.

So you were right when you said that when we go beyond abelian surfaces to higher-dimensional abelian varieties, the math would be more like QM than SR."

yes, that's more or less what i was getting at. i wasn't explicitly connecting it with jordan algebras, but i was vaguely connecting it with [the convex geometry of density matrixes and how that relates to "measurement" in qm]. (i'm vaguely imagining that the positive cone here is something like the "unnormalized density matrixes" and that the principal polarization helps you to normalize them, or something like that.) and jordan (who may have been a bit of a moron) was perhaps under the impression that they were engaged in trying to separate out the "measurement" part of qm (the jordan product) from the "dynamics" part (the lie bracket) ....

by the way, one of the mini-projects i'd really like to clarify here is how to explicitly take one of those neron-severi jordan-self-adjoints and explicitly turn it into a textbook "riemann form"! assuming that that makes sense ....

....
[Quoted text hidden]

John Baez <john.baez@ucr.edu> Tue, Mar 1, 2022 at 12:45 PM
Reply-To: baez@math.ucr.edu
To: JAMES DOLAN <james.dolan1@students.mq.edu.au>
Cc: John Baez <baez@math.ucr.edu>

Hi -

> there's a recurring theme about defining a cone of "positives" in terms of squaring (and/or some mutant form of "squaring" such as multiplying an element by its adjoint, which is hard to tell from squaring if everything's self-adjoint) wrt some formally real commutative (though not necessarily associative) multiplication.

I wrote a paper called "Getting to the bottom of Noether's theorem" which forced me to learn a lot about Jordan algebras. Faraut and Koryani's book *Analysis on Symmetric Cones* explains the correspondence between formally real Jordan algebras and self-dual cones in Euclidean spaces that are "symmetric", i.e. admit a group of linear transformations preserving the cone that's big enough to be transitive. (Then it studies differential operators on these cones, which I haven't managed to become interested in.) But to get a formally real Jordan algebra from such a cone, you need to pick an arbitrary element of the cone's interior to serve as the unit. So if they'd bothered to state the theorem as an equivalence of categories, they'd either need to talk about "cones with unit" or, better, something like quadratic Jordan algebras or Jordan triple systems.

> for example if you do this to the commutative multiplication in a formally real poisson algebra then you recover the cone of positive legendrian submanifolds in some sense (i forget the details, but it works! it's another one of those "post-graduate re-interpretation of freshman-calculus minimization-of-quadratic-polynomials techniques" ideas); furthermore you can almost recover the commutative multiplication just from knowledge of that cone, except you sort-of "lose track of the multiplicative identity-element"; in the poisson case this corresponds to the fact that you can recover a contact manifold though not a symplectic manifold just from knowing the cone of positives inside the "infinitesimal (but infinite-dimensional!) homogeneous space of legendrian submanifolds", or something like that.

Hmm, I don't know this stuff. I don't know what a "positive" Legendrian submanifold is, or how you could add them. But it sounds interesting.

> jordan (who may have been a bit of a moron) was perhaps under the impression that they were engaged in trying to separate out the "measurement" part of qm (the jordan product) from the "dynamics" part (the lie bracket) ....

Maybe I'm a bit of a moron too, since that's part of the point of my paper "Getting to the bottom of Noether's theorem". One interesting thing is that among the formally real Jordan algebras, only the self-adjoint *complex* matrices have both a Jordan and Lie structure that are compatible in a certain way (e.g. the Lie algebra acts faithfully as Jordan algebra derivations). This ability to reinterpret "observables" as "symmetry generators" is fundamental in ordinary complex mechanics, but missing in, say, real or quaternionic quantum mechanics.

> anyway, if you're looking for a conceptual interpretation of the jordan product of neron-severi elements, the above means that you can try to weasel out of it by saying that the jordan product is mostly just encoding the cone of ample elements together with a choice of favorite principal polarization to serve as the jordan identity-element (i think).

Right.

> however you could also try to _not_ weasel out of it, in which case there are some pretty speculative possibilities to explore! like, i think i see some vague possibility here of groupoidifying the jordan product into some sort of groupoid-span involving "hecke isogenies of abelian varieties" or something like that, vaguely analogous to how the lie bracket in a "hall lie-algebra" gets groupoidified into a groupoid-span involving short exact sequences of indecomposable quiver representations .... i did say that it's speculative!)

Yes, there's got to be *some* categorified meaning of the Jordan product - like some operation on line bundles, or polarizations, or something. I guess I'd better figure out if the Jordan product of two guys in the Neron-Severi lattice is again in that lattice, or just in the rationalization of that lattice. If the latter, we may need some groupoid-span trick to account for how the Jordan product of two "things" is not a "thing" but merely a "linear combination of things".

> by the way, one of the mini-projects i'd really like to clarify here is how to explicitly take one of those neron-severi jordan-self-adjoints and explicitly turn it into a textbook "riemann form"! assuming that that makes sense ....

Yes, that'd be great. I don't understand the correspondence as well as I want.

Best,
jb

JAMES DOLAN <james.dolan1@students.mq.edu.au> Tue, Mar 1, 2022 at 10:36 PM
To: John Baez <baez@math.ucr.edu>

i wrote:

"jordan (who may have been a bit of a moron) was perhaps under the impression that they were engaged in trying to separate out the "measurement" part of qm (the jordan product) from the "dynamics" part (the lie bracket) ...."

you replied:

"Maybe I'm a bit of a moron too, since that's part of the point of my paper "Getting to the bottom of Noether's theorem"."

some wikipedia excerpts:

"In 1966, Jordan published his 182-page work Die Expansion der Erde. Folgerungen aus der Diracschen Gravitationshypothese (The expansion of the Earth. Conclusions from the Dirac gravitation hypothesis)[6] in which he developed his theory that, according to Paul Dirac's hypothesis of a steady weakening of gravitation throughout the history of the universe, the Earth may have swollen to its current size, from an initial ball of a diameter of only about 7,000 kilometres (4,300 mi). This theory could explain why the ductile lower sima layer of the Earth's crust is of a comparatively uniform thickness, while the brittle upper sial layer of the Earth's crust had broken apart into the main continental plates. The continents having to adapt to the ever flatter surface of the growing ball, the mountain ranges on the Earth's surface would, in the course of that, have come into being as constricted folds.[7] Despite the energy Jordan invested in the expanding Earth theory, his geological work was never taken seriously by either physicists or geologists.[8]"

"In 1933, Jordan joined the Nazi party, like Philipp Lenard and Johannes Stark, and, moreover, joined an SA unit. He supported the Nazis' nationalism and anti-communism but at the same time, he remained "a defender of Einstein" and other Jewish scientists. Jordan seemed to hope that he could influence the new regime; one of his projects was attempting to convince the Nazis that modern physics developed as represented by Einstein and especially the new Copenhagen brand of quantum theory could be the antidote to the "materialism of the Bolsheviks". However, while the Nazis appreciated his support for them, his continued support for Jewish scientists and their theories led him to be regarded as politically unreliable.[9][10]

Jordan enlisted in the Luftwaffe in 1939 and worked as a weather analyst at the Peenemünde rocket center, for a while. During the war he attempted to interest the Nazi party in various schemes for advanced weapons. His suggestions were ignored because he was considered "politically unreliable", probably because of his past associations with Jews (in particular: Courant, Born, and Wolfgang Pauli) and the so-called "Jewish physics"."

from the talk page:

"The sentence "Had Jordan not joined the Nazi party, it is conceivable that he could have shared the 1954 Nobel Prize in Physics awarded to Max Born" is an admission of the extremely political nature of the Nobel Prize.Lestrade (talk) 14:31, 14 October 2009 (UTC)Lestrade

No, it is an admission of the extremely repulsive nature of the Nazi party. Jordans work was part sheer brilliance, part sheer crackpottery, and given that the Nobel is intented to be given to those scientists whose work conferred the "greatest benefit on mankind" (Nobel's own words) a guy who tried to make the Reich a nuclear power does not exactly qualify."

maybe "crackpottery" is meant to apply more to the expanding-earth theory than to quantum field theory or to their work on general relativity?

i've heard that people like born never forgave jordan.

i've also heard that "quantum field theory" was more or less jordan's invention. i still suspect that there are horrible level-slips and/or other conceptual mistakes embedded in quantum field theory or at least in the ways that it's traditionally taught, so the idea that it may have been invented by a nazi crackpot wouldn't startle me that much.

anyway, i'm pretty sure you already know about most of this, so the idea that jordan might have been a moron in some ways doesn't seem that controversial.

....
[Quoted text hidden]

JAMES DOLAN <james.dolan1@students.mq.edu.au>
To: John Baez <baez@math.ucr.edu> Tue, Mar 1, 2022 at 10:43 PM

"Hmm, I don't know this stuff. I don't know what a "positive" Legendrian submanifold is, or how you could add them. But it sounds interesting."

i'll try to say some more about this when i'm more awake than i am right now ....

....
[Quoted text hidden]

JAMES DOLAN <james.dolan1@students.mq.edu.au>
To: John Baez <baez@math.ucr.edu> Tue, Mar 1, 2022 at 11:11 PM

"It turns out my guess about Jordan algebras is not only right but known."

you probably already know that my philosophy about this sort of thing is that when you keep finding out that the things you're figuring out are already known it can be a sign that you're on the right track; it just takes a certain amount of patience staying on the track .... vs some contrasting philosophy that says that getting to the research frontier as quickly as possible is the main goal ....

....
[Quoted text hidden]

John Baez <john.baez@ucr.edu> Tue, Mar 1, 2022 at 11:47 PM
Reply-To: baez@math.ucr.edu
To: JAMES DOLAN <james.dolan1@students.mq.edu.au>
Cc: John Baez <baez@math.ucr.edu>

Hi -

> anyway, i'm pretty sure you already know about most of this, so the idea that jordan might have been a moron in some ways doesn't seem that controversial.

I hadn't actually known that stuff - interesting! I thought you were saying Jordan was a moron in some ways because he was trying to separate out the "measurement" and "dynamical" aspects of quantum mechanics! A number of people have tried to do that subsequently. (I didn't know he was trying to do that.)

Best,
jb

JAMES DOLAN <james.dolan1@students.mq.edu.au> Wed, Mar 2, 2022 at 9:30 AM
To: John Baez <baez@math.ucr.edu>

"I hadn't actually known that stuff - interesting! I thought you were saying Jordan was a moron in some ways because he was trying to separate out the "measurement" and "dynamical" aspects of quantum mechanics! A number of people have tried to do that subsequently. (I didn't know he was trying to do that.)"

jordan really does seem to have been an interestingly mixed bag ....

eventually there's probably more to say about "measurement" (and/or "observation") vs "dynamics" in quantum mechanics and in related contexts ....

....
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