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John Baez <johnb@ucr.edu>

paracompact hyperbolic coxeter groups

14 messages

JAMES DOLAN <james.dolan1@students.mq.edu.au> Tue, Mar 8, 2022 at 1:16 PM
To: John Baez <baez@math.ucr.edu>

hmm, this concept of "paracompact" does seem to be related to the "sporadic pattern" of "attaching one extra vanilla edge to an affine coxeter diagram corresponding roughly to a number ring D to get a paracompact hyperbolic coxeter diagram corresponding roughly to gl(2,D)" ....

also interesting the way paracompact hyperbolic coxeter groups can't seem to get very big; but it still seems like there should be some more general sort of tiling phenomenon that might allow "larger" examples ....

JAMES DOLAN <james.dolan1@students.mq.edu.au> Mon, Mar 14, 2022 at 10:56 PM
To: John Baez <baez@math.ucr.edu>

me: "also interesting the way paracompact hyperbolic coxeter groups can't seem to get very big; but it still seems like there should be some more general sort of tiling phenomenon that might allow "larger" examples ...."

more specifically, wikipedia says "The highest paracompact hyperbolic Coxeter group is rank 10". which seems interesting in light of your suggestion to look for some 9-dimensional "pgl(2,octonions)"-flavored example ....

....
[Quoted text hidden]

John Baez <john.baez@ucr.edu> Mon, Mar 14, 2022 at 11:08 PM
Reply-To: baez@math.ucr.edu
To: JAMES DOLAN <james.dolan1@students.mq.edu.au>
Cc: John Baez <baez@math.ucr.edu>

Hi -

> me: "also interesting the way paracompact hyperbolic coxeter groups can't seem to get very big; but it still seems like there should be some more general sort of tiling phenomenon that might allow "larger" examples ...."
>
> more specifically, wikipedia says "The highest paracompact hyperbolic Coxeter group is rank 10". which seems interesting in light of your suggestion to look for some 9-dimensional "pgl(2,octonions)"-flavored example ....

There's a hyperbolic Coxeter group called E10 associated to the Euclidean Coxeter group E8, which acts on 9-dimensional hyperbolic space - the "mass hyperboloid" in the 10d Minkowski spacetime that consists of the 2x2 self-adjoint octonionic matrices.

This group E10 is a way of making precise the mysterious PGL(2,Cayley). Here "Cayley" is the Cayley integers, which form a lattice in the octonions that's a rescaled version of the E8 lattices. There's a way to think of PGL(2,octonions) as the 9+1-dimensional Lorentz group, and E10 is a discrete subgroup of that.

Best,
jb

JAMES DOLAN <james.dolan1@students.mq.edu.au>
To: John Baez <baez@math.ucr.edu> Tue, Mar 15, 2022 at 12:11 AM

so maybe you're telling me that i had the exact right idea, but a glitch in my rank arithmetic? and that when that glitch is fixed then the situation is even nicer?

but if this is really right-track, then naively i think i want this alleged "E10" to be "sharply paracompact" in the sense that it's really rank 9 affine (aka "parabolic" by some people) with an extra vanilla edge stuck on at one end; giving a hyperboloidal tiling by 9d "bubbles" whose 8d boundaries look like some sort of voronoi tilings of the cayley numbers .... if the previously established pattern survives all the way into this nonassociative regime ....

so let me see if i can find a picture of this alleged "E10" online and stare at it a bit ....

hmm, so can you tell me what the relationship is between:

1. "E10" as i see depicted as for example figure 1 at https://www.semanticscholar.org/paper/Curvature-corrections-and-Kac%E2%80%93Moody-compatibility-Damour-Hanany/7b11e8ed622ffabba5d59a54a4d5de7f69244f5a, and:

2. "T_9 = [3^{6,2,1}]" (with a bar over the T) as i see in the lower-right corner of the table of "Hyperbolic paracompact groups" at https://en.wikipedia.org/wiki/Coxeter%E2%80%93Dynkin_diagram#Ranks_4%E2%80%9310

?

also this E10/T9/whatever thing _does_ seem to be "rank 9 affine with an extra vanilla edge stuck on at one end" ....

so hmmm, is it standard lore then that "E9" is the (or at least a) "affine extension" of E8, and that the E9 affine coxeter tiling is essentially just the voronoi tiling for the cayley integers ???? hmmm. i guess you already sort-of confirmed that ....

i might as well also ask whether there might be some complex projective surface whose neron-severi group we should try to conflate with the E8 lattice, given that we've idly thought about trying to relate "spin factor" jordan algebras to neron-severi groups of general-ish complex projective surfaces ....

also, i guess i skipped past the quaternions to the octonions here, so at some point should go back and look at the quaternions ....

....
[Quoted text hidden]

JAMES DOLAN <james.dolan1@students.mq.edu.au>
To: John Baez <baez@math.ucr.edu> Tue, Mar 15, 2022 at 12:23 AM

hmm, but i don't see any mention of that E10/T9/whatever thing at https://en.wikipedia.org/wiki/Paracompact_uniform_honeycombs so far ....

that seems annoying ....

....
[Quoted text hidden]

JAMES DOLAN <james.dolan1@students.mq.edu.au>
To: John Baez <baez@math.ucr.edu> Tue, Mar 15, 2022 at 12:26 AM

me:

"hmm, but i don't see any mention of that E10/T9/whatever thing at https://en.wikipedia.org/wiki/Paracompact_uniform_honeycombs so far ....

that seems annoying ...."

or is that wikipedia article maybe restricted just to 3-dimensional / rank-4 examples ?? ....

....
[Quoted text hidden]

JAMES DOLAN <james.dolan1@students.mq.edu.au> Tue, Mar 15, 2022 at 12:38 AM
To: John Baez <baez@math.ucr.edu>

offhand i don't think we've ruled out yet the possibility of there being other integer forms of the octonions besides the cayley integers, which might give other "sharply paracompact rank 10 coxeter tilings", if any such exist ....

....
[Quoted text hidden]

JAMES DOLAN <james.dolan1@students.mq.edu.au> Tue, Mar 15, 2022 at 3:06 AM
To: John Baez <baez@math.ucr.edu>

hmm ....

i mentioned these 2 links:

https://www.semanticscholar.org/paper/Curvature-corrections-and-Kac%E2%80%93Moody-compatibility-Damour-Hanany/7b11e8ed622ffabba5d59a54a4d5de7f69244f5a

and:

https://en.wikipedia.org/wiki/Coxeter%E2%80%93Dynkin_diagram#Ranks_4%E2%80%9310

and i mentioned how they both seemed to be talking about the same coxeter group though under different names ("E10" and some messier name involving "T" and "9") ....

however i just noticed that they actually seem to have more in common than that .... figures 3, 4, and 1 from the first link seem to correspond pretty exactly to the last 3 entries in the bottom row of the table from the second link ....

so some stuff is fitting together, apparently ....

....
On Tue, Mar 15, 2022 at 3:11 AM JAMES DOLAN <james.dolan1@students.mq.edu.au> wrote:
[Quoted text hidden]

John Baez <john.baez@ucr.edu> Tue, Mar 15, 2022 at 10:30 AM
Reply-To: baez@math.ucr.edu
To: JAMES DOLAN <james.dolan1@students.mq.edu.au>
Cc: John Baez <baez@math.ucr.edu>

Hi -

> so maybe you're telling me that i had the exact right idea, but a glitch in my rank arithmetic? and that when that glitch is fixed then the situation is even nicer?

Which glitch? Rank 10 hyperbolic Coxeter groups should act on 10d Minkowski spacetime and thus the 9d hyperboloid in there. Is that what you mean?

> but if this is really right-track, then naively i think i want this alleged "E10" to be "sharply paracompact" in the sense that it's really rank 9 affine (aka "parabolic" by some people) with an extra vanilla edge stuck on at one end; giving a hyperboloidal tiling by 9d "bubbles" whose 8d boundaries look like some sort of voronoi tilings of the cayley

> numbers .... if the previously established pattern survives all the way into this nonassociative regime ....

I'm hoping that's true. I wrote two blog articles on this stuff - the first two here:

https://math.ucr.edu/home/baez/octonions/integers/

But then I veered off into some other directions, not trying to get a good picture of the 9d hyperbolic honeycomb. So I don't know if it's paracompact.

> so let me see if i can find a picture of this alleged "E10" online and stare at it a bit ....

I guess you found it; it's also in my blog articles.

> hmm, so can you tell me what the relationship is between:
>
> 1. "E10" as i see depicted as for example figure 1 at https://www.semanticscholar.org/paper/Curvature-corrections-and-Kac%E2%80%93Moody-compatibility-Damour-Hanany/7b11e8ed622ffabba5d59a54a4d5de7f69244f5a, and:
>
> 2. "T_9 = [3^{6,2,1}]" (with a bar over the T) as i see in the lower-right corner of the table of "Hyperbolic paracompact groups" at https://en.wikipedia.org/wiki/Coxeter%E2%80%93Dynkin_diagram#Ranks_4%E2%80%9310

The Coxeter diagram for "T_9" is exactly the same as the E10 Coxeter diagram, so I guess it's just another name for the same thing!

> also this E10/T9/whatever thing _does_ seem to be "rank 9 affine with an extra vanilla edge stuck on at one end" ....

Yes.

> so hmmm, is it standard lore then that "E9" is the (or at least a) "affine extension" of E8, and that the E9 affine coxeter tiling is essentially just the voronoi tiling for the cayley integers ???? hmmm. i guess you already sort-of confirmed that ....

Yes, that's right, at least if I don't worry about the difference between the Voronoi tiling for the Cayley integers and its dual.

> i might as well also ask whether there might be some complex projective surface whose neron-severi group we should try to conflate with the E8 lattice, given that we've idly thought about trying to relate "spin factor" jordan algebras to neron-severi groups of general-ish complex projective surfaces ....

I just looked around for that last night and found it! I should have thought about this sooner:

Arnaud Beauville
Abelian varieties associated to Gaussian lattices
https://arxiv.org/abs/1112.2843

We associate to a unimodular lattice Γ, endowed with an automorphism i of square -1, a principally polarized abelian variety A_Γ.
We show that the configuration of i-invariant theta divisors of A_Γ follows a pattern very similar to the classical theory of theta characteristics; as a consequence we find that A_Γ has a high number of vanishing thetanulls. When $\Gamma = E_8$ we recover the 10 vanishing thetanulls of the abelian fourfold discovered by R. Varley.

So we're getting a very nice abelian fourfold! The paper explains the idea of a "vanishing thetanull:

> it has 10 “vanishing thetanulls” (even theta functions vanishing at 0), the maximum possible for a 4-dimensional indecomposable principally polarized abelian variety.

There should be tons of cool math relating this abelian fourfold to other great things mentioned in my blog articles!

> also, i guess i skipped past the quaternions to the octonions here, so at some point should go back and look at the quaternions ....

Right! If you ever want to "cheat", these are discussed here:

Daniel Allcock
New complex- and quaternion-hyperbolic reflection groups
https://arxiv.org/abs/math/9907193

> We consider the automorphism groups of various Lorentzian lattices over the Eisenstein, Gaussian, and Hurwitz integers, and in some of them we find reflection groups of finite index. These provide new finite-covolume reflection groups acting on complex and quaternionic hyperbolic spaces. Specifically, we provide groups acting on CH^n for all n<6 and n=7, and on HH^n for n=1,2,3 and 5. We compare our groups to those discovered by Deligne and Mostow and show that our largest examples are new. For many of these Lorentzian lattices we show that the entire symmetry group is generated by reflections, and obtain a description of the group in terms of the combinatorics of a lower-dimensional positive-definite lattice. The techniques needed for our lower-dimensional examples are elementary, but to construct our best examples we also need certain facts about the Leech lattice. We give a new and geometric proof of the classifications of selfdual Eisenstein lattices of dimension < 7 and of selfdual Hurwitz lattices of dimension < 5.

and here:

Norman W. Johnson and Asia Ivić Weiss
Quadratic integers and Coxeter groups
https://www.cambridge.org/core/journals/canadian-journal-of-mathematics/article/quadratic-integers-and-coxeter-groups/CF262D475903A0104145D1294DA80EF9

> Matrices whose entries belong to certain rings of algebraic integers can be associated with discrete groups of
>
> transformations of inversive n -space or hyperbolic $(n+1)-\mathrm{space}\ \mathrm{H}^{n+1}$. For small n , these may be Coxeter
>
> groups, generated by reflections, or certain subgroups whose generators include direct isometries of H^{n+1} . We show how linear fractional transformations over rings of rational and (real or imaginary) quadratic integers are related to the symmetry groups of regular tilings of the hyperbolic plane or 3-space. New light is shed on the
>
> properties of the rational modular group $\mathrm{PSL}_2(\mathrm{Z})$, the Gaussian modular (Picard) group $\mathrm{PSL}_2(\mathrm{Z}[i])$, and the
>
> Eisenstein modular group $\mathrm{PSL}_2(\mathrm{Z}[\omega])$.

Sorry for crappy formatting here!

Best,
jb

JAMES DOLAN <james.dolan1@students.mq.edu.au> Tue, Mar 15, 2022 at 10:51 AM
To: John Baez <baez@math.ucr.edu>

you: "Which glitch? Rank 10 hyperbolic Coxeter groups should act on 10d Minkowski spacetime and thus the 9d hyperboloid in there. Is that what you mean?"

yes, i think so. i think i was just making some sort of fencepost mistake between coxeter "rank" (= number of dots in coxeter diagram i guess) and dimension of coxeter tiling.

[Quoted text hidden]

JAMES DOLAN <james.dolan1@students.mq.edu.au> Tue, Mar 15, 2022 at 10:58 AM
To: John Baez <baez@math.ucr.edu>

seems a bit funny if we're able to morally identify the element

" 0 1
1 0 "

in "pgl(2, cayley integers)", given the slapdashness of "pgl(2)" and/or of matrix notation in the nonassociative case ....

....

[Quoted text hidden]

JAMES DOLAN <james.dolan1@students.mq.edu.au> Tue, Mar 15, 2022 at 11:04 AM
To: John Baez <baez@math.ucr.edu>

i need to think more about this "vanishing thetanulls" stuff ....

however i'm still wondering whether there might be some surface inside the abelian 4-fold, whose neron-severi group fits in here somehow ....

....

On Tue, Mar 15, 2022 at 1:31 PM John Baez <john.baez@ucr.edu> wrote:

[Quoted text hidden]

John Baez <john.baez@ucr.edu> Tue, Mar 15, 2022 at 11:29 AM
Reply-To: baez@math.ucr.edu
To: JAMES DOLAN <james.dolan1@students.mq.edu.au>
Cc: John Baez <baez@math.ucr.edu>

Hi -

On Tue, Mar 15, 2022 at 11:05 AM JAMES DOLAN <james.dolan1@students.mq.edu.au> wrote:

> however i'm still wondering whether there might be some surface inside the abelian 4-fold, whose neron-severi group fits in here somehow ....

Yeah, I kinda forgot for a bit that the Neron-Severi group of an abelian 4-fold should probably be sitting in a Jordan algebra with a quartic form on it, not a quadratic form! So all this stuff about 10d Minkowski spacetime is a bit mysterious. But a suitable abelian surface might explain the existence of that quadratic form.

Best,
jb

JAMES DOLAN <james.dolan1@students.mq.edu.au> Tue, Mar 22, 2022 at 8:49 AM
To: John Baez <baez@math.ucr.edu>

there's a curiosity here that i'm not sure what to make of yet ....

the idea is something like this:

what if we try to do for [the paracompact hyperbolic tilings associated with integer forms of the complex numbers / quaternions / octonions] the same thing that we do for [the paracompact hyperbolic tiling associated with "the integer form of the reals"] whenever we consider modular curves?

so for example, consider the group homomorphism psl(2,Z) -> psl(2,Z/5), and consider how this relates to wrapping the (infinity,3) coxeter-tiling around the dodecahedron (or something like that), as studied in connection with modular curves.

but then analogously consider for example the group homomorphism psl(2,gaussian integers]) -> psl(2,gaussian integers / 2i+1), and consider how it relates to wrapping the (4,4,3) coxeter-tiling around a tiling where the tiles are solid toruses whose plane-torus boundaries look like a voronoi tiling of the gaussian integers modulo 2i+1 .... and so forth.

("solid torus whose plane-torus boundary looks like a voronoi tiling of the gaussian integers modulo 2i+1" is supposed to be analogous here to a pentagonal face of the dodecahedron; the boundary of a pentagon looks like a voronoi tiling of Z/5.)

is there something analogous to modular curves that comes into play here??

not sure how well i explained what i'm trying to get at here, but you can probably guess what i mean just by thinking about it. or i could try to explain it to you again when we get a chance to talk ....

....
[Quoted text hidden]