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John Baez <johnb@ucr.edu>

underlying real affine-algebraic group of a complex affine-algebraic group

5 messages

JAMES DOLAN <james.dolan1@students.mq.edu.au>
To: john.baez@ucr.edu

Fri, Mar 17, 2023 at 2:04 AM

so i should make an attempt here to clarify some of the ideas that i was trying to describe towards the end of the discussion today-- roughly speaking, general abstract ideas about the interplay between contrasting 2-rigs (conceived of as syntactic categories of theories ....) arising from the representations of "the same" affine-algebraic group g, but treated as affine-algebraic wrt two different commutative rings k1 and k2 with a homomorphism k1 -h-> k2; thus for example h might be the inclusion of k1 := the real numbers into k2 := the complex numbers, and g might be gl(1,k2), which is a perfectly good complex-affine-algebraic group but which we can also choose to view as a _real_-affine-algebraic group instead.

(meanwhile all of this general abstract stuff is allegedly just something to apply to the particular concrete situation that we're interested in, involving z/3-torsors over number fields. this is perhaps a big general abstract digression to chew on in pursuit of a relatively humble particular concrete problem, but that's the way it is at the moment!)

thus we're now in a position to clarify some issues that have caused me a lot of conceptual confusion in the past, like: what's the relationship between the complex-linear (aka "holomorphic") representations of g := gl(1, the complex numbers) vs the real-linear representations of that same group g? and the flavor of answer that we're probably going to get is something like this: those complex-linear representations are essentially the co-modules of a certain complex commutative hopf algebra and accordingly form the syntactic category of a certain theory; meanwhile those real-linear representations are essentially the co-modules of a certain real commutative hopf algebra and accordingly form the syntactic category of a certain different theory; and these two theories co-exist in the same doctrine where we can contemplate theory-interpretations between them arising in a conceptual way from the inclusion homomorphism h from the real numbers into the complex numbers.

(as i'm lying here writing this, i'm trying to convince myself not to be horrified by the conceptual complexity of the "debauch of base-changes" here. thus i'm trying to convince myself that the syntactic-category-of-a-theory level of thinking is the royal road that acts as a good conceptual guide to disentangling the "debauch of base-changes" that occurs at the commutative-hopf-algebra level of thinking ....)

so let me continue thinking outloud here for a bit about this example of gl(1, the complex numbers) as an affine-algebraic group over the real numbers vs as an affine-algebraic group over the complex numbers, and about the interplay between the two corresponding comodule-categories construed as theories co-existing in the same doctrine ....

so let's do a bit of explicit calculation (presumably supplemented by some judicious guessing ....) ....

so the coordinate C-algebra of the underlying complex affine variety of the complex affine-algebraic group gl(1,C) is C[x,y]/xy=1. on top of that commutative C-algebra we have a co-multiplication making it into a commutative hopf C-algebra about which we can make some judicious guesses.

(here i'm reduced to using capital C as an abbreviation for "the complex numbers", for the moment ....)

meanwhile the coordinate R-algebra of the underlying real affine variety of the real affine-algebraic group gl(1,C) is R[a,b,c,d]/"(a+bi)*(c+di)=1+0i". on top of that we have a co-multiplication making it into a commutative hopf R-algebra about which we can make some judicious guesses.

so the comodule-category of the commutative hopf C-algebra here is essentially a complex 2-vector space with basis indexed by the integers, and the tensor product of those co-modules corresponds to the addition of those integer indexes ....

but what is the comodule-category of the commutative hopf R-algebra here, and what is the tensor-product on it like?? we should be able to judiciously guess at this, and/or find ourselves reduced to performing explicit calculations to figure it out. after that we can think about what kind of theory-interpretations we should get between these two comodule-categories. "should" here means on the basis of good conceptual "common-sense"-like thinking ....

well, i'm going to stop writing this for now. of course we have a lot of options here: keep pursuing this example in detail, or perhaps switch to different examples where the role of the ("base-change") inclusion homomorphism from the real numbers into the complex numbers is usurped by the inclusion of the real numbers into study's "dual numbers", and/or by the diagonal inclusion of the real numbers into the pairs of real numbers, and/or by the inclusion of the rational numbers into the eisenstein numbers (getting somewhat close to the original motivating problem we're _supposed_ to be thinking about!), and/or etc. or we could retreat and try some different tack entirely ....

JAMES DOLAN <james.dolan1@students.mq.edu.au> Fri, Mar 17, 2023 at 2:15 AM
To: john.baez@ucr.edu

after the discussion on thursday afternoon i began to worry that i had hallucinated the idea that study was a german mathematician who had something to do with the dual numbers (and thus that i'd look silly pronouncing it "SHTOO-dee"). wikipedia does seem to participate in the same hallucination though ....

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JAMES DOLAN <james.dolan1@students.mq.edu.au> Fri, Mar 17, 2023 at 2:38 AM
To: john.baez@ucr.edu

hmm, in study's wikipedia article there's also an interesting bit about study having acted as a skeptic towards some work of sylvester and gordan suggesting (more or less prior to planck's work) somewhat speculative analogies between invariant theory and chemical valence theory, work which was later vindicated by developments in quantum chemistry.

my main excuse for mentioning this here is to remind myself to try to learn more about this, including trying to place it in the same context with kummer's analogy between "radicals" in number theory and in chemistry-- partially because i have a vague feeling that in the pre-history of my fascination with kummer's analogy, arising from having read about it in some expository writing somewhere when i was very young, they also mentioned the sylvester/gordan work and its later vindication.

here's the "Valence theory" section of study's wikipedia article:

"Somewhat surprisingly Eduard Study is known by practitioners of quantum chemistry. Like James Joseph Sylvester, Paul Gordan believed that invariant theory could contribute to the understanding of chemical valence. In 1900 Gordan and his student G. Alexejeff contributed an article on an analogy between the coupling problem for angular momenta and their work on invariant theory to the Zeitschrift für Physikalische Chemie (v. 35, p. 610). In 2006 Wormer and Paldus summarized Study's role as follows:[10]

"The analogy, lacking a physical basis at the time, was criticised heavily by the mathematician E. Study and ignored completely by the chemistry community of the 1890s. After the advent of quantum mechanics it became clear, however, that chemical valences arise from electron–spin couplings ... and that electron spin functions are, in fact, binary forms of the type studied by Gordan and Clebsch.""

....
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JAMES DOLAN <james.dolan1@students.mq.edu.au> Fri, Mar 17, 2023 at 5:04 AM
To: john.baez@ucr.edu

so i'm going to try continuing a bit here to pursue the theme of "how a base-change k1 -h-> k2 between commutative rings together with a commutative hopf k2-algebra g manifests at the level of comodule-categories (construed as syntactic categories of theories)", not necessarily directly following up on previous examples but trying to describe a network of theory-interpretations arising in whole or in part from h together with g.

so first of all, each commutative ring k covariantly gives rise to "the theory of a tensor-monoid homomorphism from k to the tensor-unit", whose syntactic category (as you can semi-pleasantly calculate) is the module category of k. thus the theory interpretation induced by h takes [the homomorphism from k1 to the tensor-unit] to [the homomorphism from k2 to the tensor-unit, pre-composed by h]. (and if you play around with it a bit you'll probably see that this abstract nonsense should resonate with all sorts of more concrete stuff that you already know. for example, roughly speaking this shows that "the concept of spectrum of a 2-rig is a generalization of the concept of spectrum of a ring" ....)

second of all, each commutative hopf k-algebra g covariantly gives rise to "the theory of a g-torsor", whose syntactic category is the comodule category of g. in the case where g is the initial commutative hopf k-algebra, the g-comodules coincide with the k-modules; thus we automatically have a theory-interpretation from the theory associated to the commutative ring k to the theory associated to any commutative hopf k-algebra g.

third of all, commutative hopf algebras can be transported along commutative ring homomorphisms-- we'll have to think about whether they can be transported not only covariantly but perhaps also contravariantly, perhaps by means of some adjointness phenomenon! this will contribute to the network of theory-interpretations arising from a commutative hopf algebra together with some sort of base-change ....

that's it for now; maybe i'll figure out more stuff to write later. i'm writing pretty extemporaneously in these recent emails so i don't have a good sense yet of how much sense this might be making for you; perhaps it's mostly just helping me to try to organize my thoughts ....

in any case i'm really hoping to get back to showing how this kind of stuff should manifest in the context of torsors over number fields !! ....

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On Fri, Mar 17, 2023 at 5:04 AM JAMES DOLAN <james.dolan1@students.mq.edu.au> wrote:
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John Baez <john.baez@ucr.edu> Sat, Mar 25, 2023 at 9:55 AM
Reply-To: baez@math.ucr.edu
To: JAMES DOLAN <james.dolan1@students.mq.edu.au>

Hi -

I haven't been answering these emails because I'm not quite sure where you're going with all this. But if we get into how a given complex affine-algebraic group has distinct real forms, or similar things for other field-subfield pairs, we're bound to bump into "Galois descent", which is the usual way people classify these "forms". So maybe I really *should* talk about Galois descent sometime soon.

As usual, it's a lot less complicated and more beautiful than most treatments make it sound. I worked a lot to understand it when I was trying to understand the Albert-Brauer-Hasse-Noether result:

Central simple division algebras over a field k are classified by H^1(G, PGL(∞,K)). Here K is the separable closure of k and G is the Galois group of K over k, aka the absolute Galois group of k.

Think of PGL(∞,K) as the automorphism group of an arbitrarily big matrix algebra over K.

So, central simple division algebras over k are like "bundles of matrix algebras over K" over the space whose fundamental group is G.

This is so cool!

There's a similar result with Hopf algebras replacing central simple division algebras.... I just bumped into it.

Best,
jb

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