THE MYSTERIES OF COUNTING —

Euler Characteristic χ

vs.

Homotopy Cardinality $|\ |$

\-  = ?

for more:

http://math.ucr.edu/home/baez/counting

Natural numbers are cardinalities of finite sets:

$+$ is disjoint union (coproduct):

$$\bullet\ \bullet\ \bullet\ +\ \bullet\ \bullet\ =\ \bullet\ \bullet\ \bullet\ \bullet\ \bullet$$

$\times$ is Cartesian product (product):

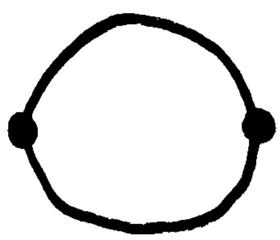
$$\begin{array}{c}\bullet\\ \bullet\end{array}\ \times\ \bullet\ \bullet\ \bullet\ =\ \begin{array}{ccc}\bullet & \bullet & \bullet\\ \bullet & \bullet & \bullet\end{array}$$

What sort of thing has cardinality -1, or $\frac{5}{2}$?

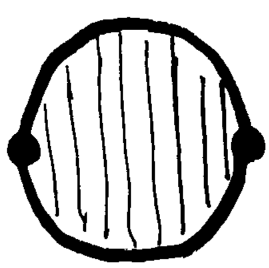
NEGATIVE SETS

2 islands

1 island –
a bridge is a
negative island!



0 islands?



1 island –
a bridge between
bridges is a
negative bridge!

EULER-SCHANUEL CHARACTERISTIC

Set

$$\chi(\circ\!\!-\!\!\circ) = -1$$

and demand compatibility with $+, \times$:

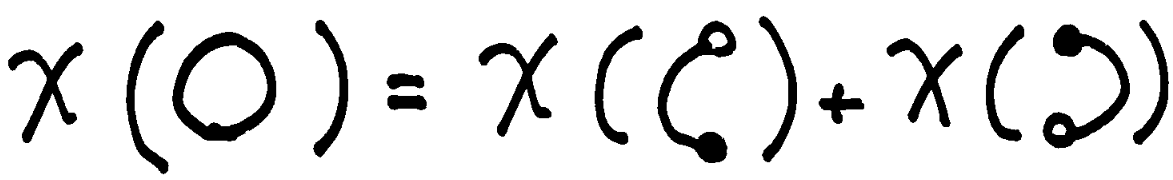
$$\chi(\bullet) = 1$$

$$\chi(\bullet\!\!-\!\!\circ) = \chi(\bullet) + \chi(\circ\!\!-\!\!\circ)$$

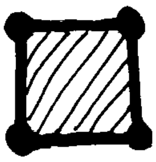
$$= 1 + -1 = 0$$

$$\chi(\bullet\!\!-\!\!\bullet) = \chi(\bullet) + \chi(\bullet\!\!-\!\!\circ)$$

$$= 1 + 0 = 1$$



$$= 0 + 0 = 0$$

$\chi($  $) = \chi($  $) \times \chi($  $)$

$= 1 \times 1 = 1$

$\chi($  $) = \chi($  $) \times \chi($  $)$

$= 1 \times -1 = -1$

$\chi($  $) = \chi($  $) \times \chi($  $)$

$= -1 \times -1 = 1$

$$\chi(\mathbb{R}^n) = (-1)^n$$

$\chi($  $) = \chi($  $)$

$+ \chi($  $)$

$= \chi(\bigcirc) \times \chi($  $) + \chi(\bigcirc) \times \chi($  $)$

$= 0 \times 1 + 0 \times -1 = 0$

Euler - Schanuel characteristic agrees with ordinary Euler characteristic on <u>compact</u> spaces:

$$\chi(X) = \text{rank}(H^0(X)) - \text{rank}(H^1(X)) + \text{rank}(H^2(X)) - \cdots$$

but in general it is defined using <u>compactly supported</u> cohomology.

Euler characteristic is defined for "cohomologically finite" spaces: those for which the sum converges.

Note: with Euler-Schanuel characteristic we have:

$$\chi(\mathbb{R}) = -1,$$
$$\chi(\mathbb{C}) = 1.$$

How is $\mathbb{R}$ like the "field with -1 elements"? How is $\mathbb{C}$ like "the field with 1 element"?

Consider the field with q elements, $\mathbb{F}_q$. The projective space

$$\mathbb{F}_q P^n = \{\text{lines in } \mathbb{F}_q^{n+1}\}$$

has

$$|\mathbb{F}_q P^n| = 1 + q + q^2 + \cdots + q^n$$

If

$$\text{``}\mathbb{R} = F_{-1}, \quad \mathbb{C} = F_1\text{''}$$

then we'd expect

$$|\mathbb{R}P^n| = 1 + (-1) + (-1)^2 + \cdots + (-1)^n$$

$$|\mathbb{C}P^n| = 1 + 1 + 1^2 + \cdots + 1^n$$

In fact we have:

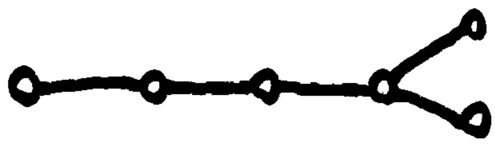
$$\chi(\mathbb{R}P^n) = 1 + (-1) + (-1)^2 + \cdots + (-1)^n$$

$$\chi(\mathbb{C}P^n) = 1 + 1 + 1^2 + \cdots + 1^n$$

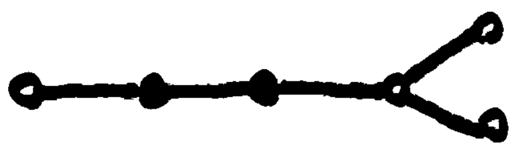
The reason: for any field F we have "Schubert cells":

$$FP^n = 1 + F + F^2 + \cdots + F^n$$

More generally: for any Dynkin diagram



we get a simple algebraic group G_F over any field F, & marking some dots:



picks out a subgroup P_F. We have:

$$|G_{F_q}/P_{F_q}| = \varphi(q)$$

for some polynomial φ, and:

$$\chi(G_{\mathbb{R}}/P_{\mathbb{R}}) = \varphi(-1),$$

$$\chi(G_{\mathbb{C}}/P_{\mathbb{C}}) = \varphi(1).$$

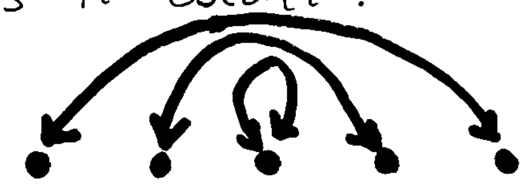
FRACTIONAL SETS

Sometimes division works nicely:

$4/2 = 2$ 

This action of $\mathbb{Z}_2$ on 4 has 2 orbits.

Sometimes it doesn't:

$5/2 = 2\frac{1}{2}$ 

This action of $\mathbb{Z}_2$ on 5 has 3 orbits.

We don't always have

$$|X/G| = |X|/|G|$$

The solution is to count the middle dot as <u>half a point</u>, since it has <u>2-fold symmetry</u>!

To do this we can define the "weak quotient" $X/\!/G$, which is a groupoid & prove

$$|X/\!/G| = |X|/|G|$$

where left side is "groupoid cardinality" – counting isomorphism classes of objects inversely weighted by the size of their symmetry group.

But, we can see groupoids as spaces with vanishing homotopy groups above the first. So, we can be even more general & work with <u>spaces</u>.

HOMOTOPY CARDINALITY

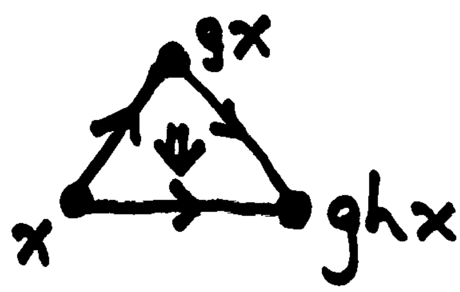
For a connected space X let

$$|X| = \frac{1}{|\pi_1(X)|} \cdot |\pi_2(X)| \cdot \frac{1}{|\pi_3(X)|} \cdots$$

Say X is "homotopically finite" if this converges. For a general space X define $|X|$ as a sum over components. Say X is "tame" if this sum converges.

Given a topological group G acting on a space X, define the "homotopy quotient" $X/\!/G$ by sewing in a <u>path</u> from x to gx for all x, $g \neq 1$:

a <u>path</u> <u>of</u> <u>paths</u> for all x, $g \neq 1$, $h \neq 1$:



and so on! We then have

$$|X + Y| = |X| + |Y|$$

$$|X \times Y| = |X| \times |Y|$$

$$|X/\!/G| = |X| / |G|$$

EXAMPLES

1)

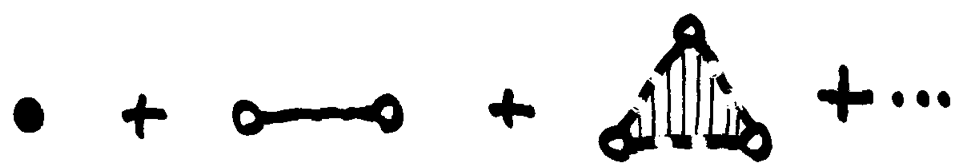
$$BG := \frac{1}{G}$$

has

$$|BG| = \frac{1}{|G|}$$

2)

$$B\mathbb{Z}_2 = \mathbb{R}P^\infty$$



has

$$\chi(\mathbb{R}P^\infty) = 1 - 1 + 1 + \cdots$$

but

$$|\mathbb{R}P^\infty| = \frac{1}{|\mathbb{Z}_2|} = \frac{1}{2}$$

3) Let E be the "space of finite sets", i.e. the space of finite subsets of $\mathbb{R}^\infty$.

We have

$$E \simeq \frac{1}{S_0} + \frac{1}{S_1} + \frac{1}{S_2} + \cdots$$

where S_n is the permutation group on n letters. So,

$$|E| = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \cdots$$

$$= e$$

EULER χ VERSUS HOMOTOPY $|\ |$

Conjecture: The only spaces that are both cohomologically & homotopically finite are finite sets of points — up to weak homotopy equivalence.

If so, it makes no sense to ask if Euler χ and homotopy $|\ |$ are "the same", except on finite sets, where they are.

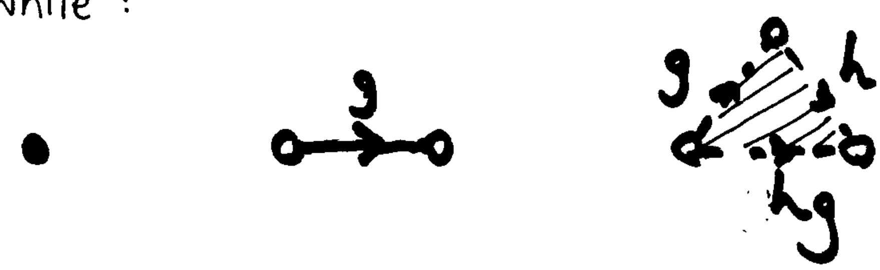
Or does it ???

EXAMPLES

1) $BG = 1/\!/G$ with G a finite group.

$$|BG| = \frac{1}{|G|}$$

while:



$$\chi(BG) = 1 - (|G|-1) + (|G|-1)^2 - \cdots$$

$$\text{``}=\text{''} \quad \frac{1}{1+(|G|-1)}$$

$$= \frac{1}{|G|} \quad !?$$

This is not crazy; we can use the "Abel sum":

$$A\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} a_n z^n \Big|_{z=1}$$

analytically continued!

James Propp has obtained many results on this generalization of χ, e.g.:

2) Let $\mathbb{R}^{[0,1]}$ be the space of piecewise-linear maps $f: [0,1] \to \mathbb{R}$.

Then

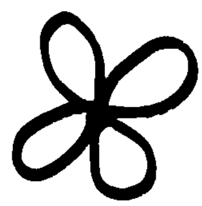
$$\chi(\mathbb{R}^{[0,1]}) = 1 + 2 + 4 + \cdots$$

$$\underset{A}{=} \frac{1}{1-2} = -1$$

so

$$\chi(\mathbb{R}^{[0,1]}) \underset{A}{=} \chi(\mathbb{R})^{\chi([0,1])}$$

3) Let F_n be the free group on n generators. Then

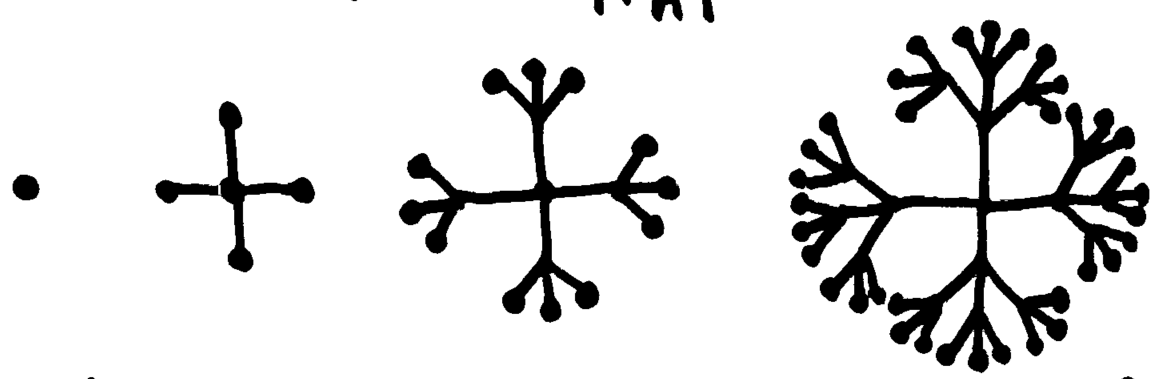
$$BF_n =$$  bouquet of n circles

so

$$\chi(BF_n) = 1-n$$

while

$$|BF_n| = \frac{1}{|F_n|}$$



$$|F_n| = 1 + 2n + 2n(2n-1) + 2n(2n-1)^2 + \cdots$$

$$\underset{A}{=} 1 + \frac{2n}{1-(2n-1)} = \frac{1}{1-n}$$

so $$|BF_n| \underset{A}{=} 1-n$$

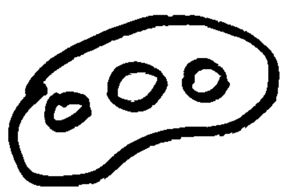
4) Floyd & Plotkin have shown:

$\chi($  $) = 2-2g$

g holes

$G = \pi_1($  $)$

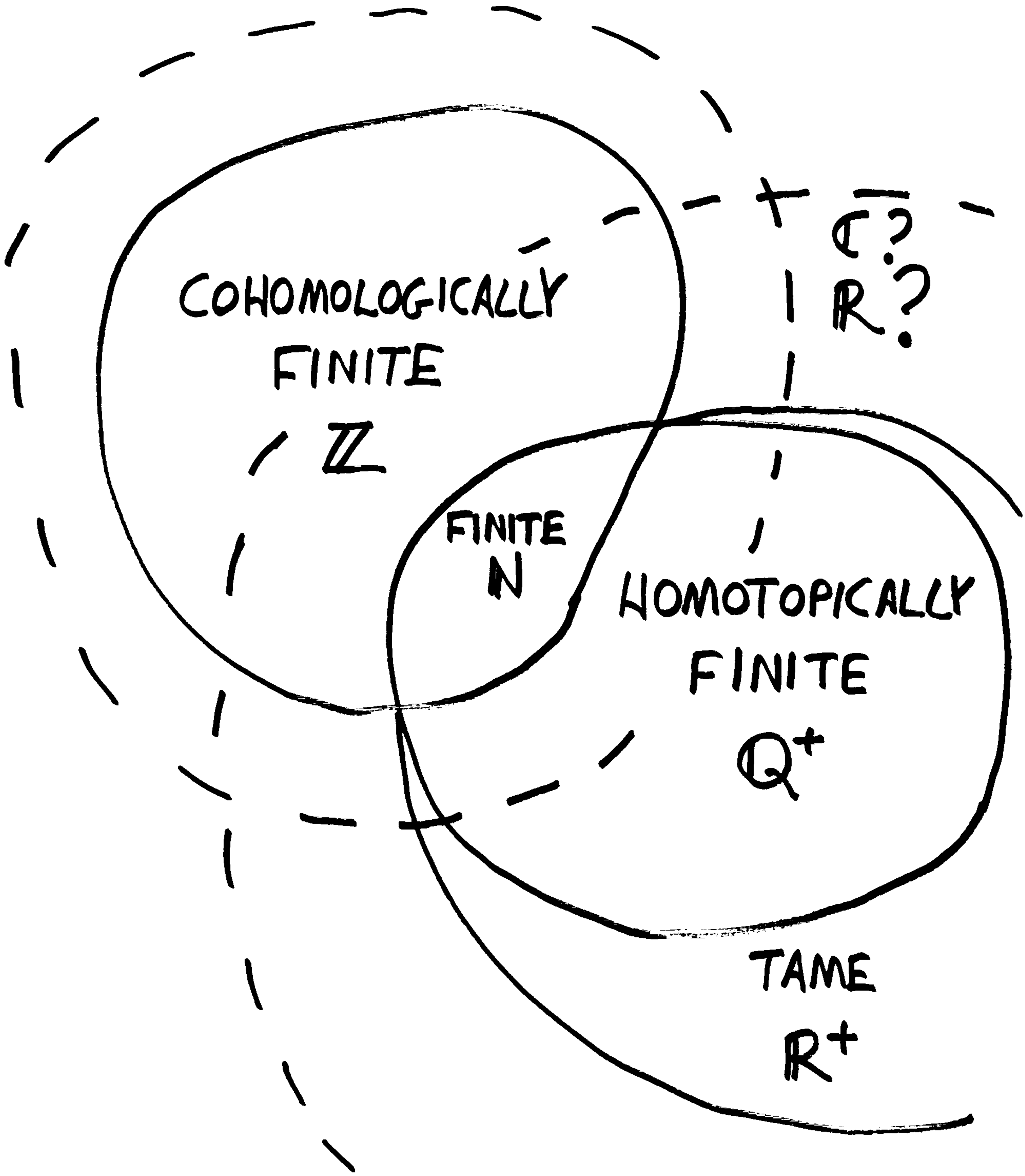
$= \langle e_i, f_i \ \ i=1,\dots,g \mid [e_1,f_1]\cdots[e_g,f_g]=1 \rangle$

 $\simeq BG$

$|G| \underset{A}{=} \dfrac{1}{2-2g}$!!

so

$|$  $| = \dfrac{1}{|G|} \underset{A}{=} 2-2g$


$\mathbb{C}$?
$\mathbb{R}$?
COHOMOLOGICALLY FINITE
$\mathbb{Z}$
FINITE
$\mathbb{N}$
HOMOTOPICALLY FINITE
$\mathbb{Q}^+$
TAME
$\mathbb{R}^+$

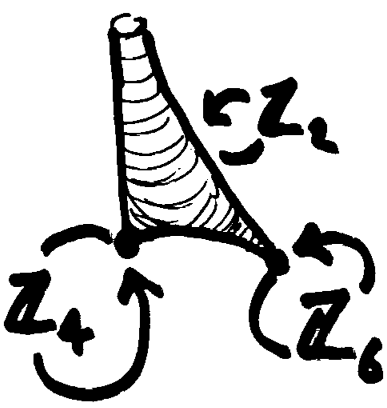
ORBIFOLD EULER χ

1) $\chi(\mathbb{R}/\!/\mathbb{Z}_2) =$

$\chi($  $) = -1 + \frac{1}{2}$

$= -\frac{1}{2}$

2) $\chi(H/\!/SL(2,\mathbb{Z})) =$

$\chi($ 

$) =$

$-\frac{1}{2} + \frac{1}{4} + \frac{1}{6} = -\frac{1}{12} = \zeta(-1)$

Massively generalized by G. Harder.

STUFF TYPES & HOMOTOPY II

$$\begin{array}{ccccc} F & = & \sum_{n=0}^{\infty} & F_n \\ \downarrow & & & \downarrow \\ E & = & \sum_{n=0}^{\infty} & 1/\!/S_n \end{array}$$

"finite sets with extra stuff"

"finite sets"

Given a space X, let $F(X)$ be the space of "X-colored F-stuffed finite sets":

$$F(X) = \sum_{n=0}^{\infty} X^n /\!/ G_n \qquad F_n = 1/\!/G_n$$

and define

$$|F(X)| = A\sum_{n=0}^{\infty} \left| X^n /\!/ G_n \right|$$

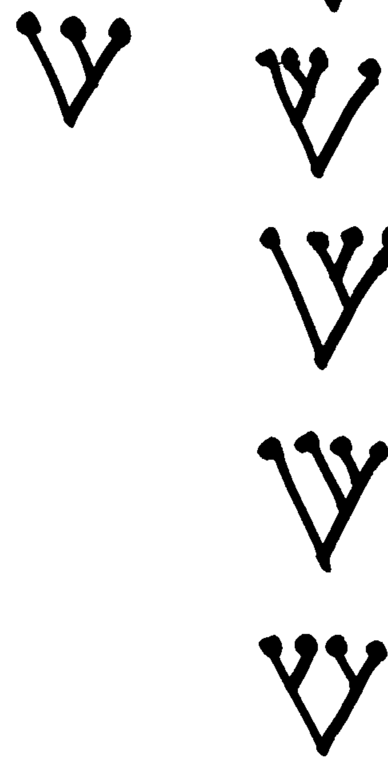
$$= A\sum_{n=0}^{\infty} |F_n|\,|X|^n$$

For example:

$$F = \text{"binary rooted planar trees"}$$

$$F(X) = X + X^2 + 2X^3 + 5X^4 + \cdots$$

$$|F(X)| = \frac{1-\sqrt{1-4|X|}}{2}$$



$$|F(\bullet)| = \frac{1-\sqrt{-3}}{2} \; !$$

Lawvere, Blass, Schanuel, Gates, Leinster, Fiore, ...

$$|F(\circ\!\!-\!\!\circ)| = \frac{1-\sqrt{5}}{2} \; !$$

Propp, Houston, ...